APPENDIX 6.2: PROOF THAT $m \pm z_{\alpha/2}(\sigma_m)$ WILL CAPTURE μ $(1-\alpha)100\%$ OF THE TIME

The purpose of this appendix is to show that $m \pm z_{\alpha/2}(\sigma_m)$ will capture μ $(1-\alpha)100\%$ of the time, no matter what the population mean and standard deviation are or what the sample size is. The only assumption here is that the distribution of means is normal.

We saw in Chapter 6 that any sampling distribution of the mean can be transformed to the z-distribution by applying the following transformation to each sample mean:

$$z = \frac{m-\mu}{\sigma_m}.$$

We then stated that $(1-\alpha)100\%$ of all possible intervals computed as

$$m \pm z_{\alpha/2}(\sigma_m)$$

will capture μ. So, let's see why this is true.

By definition we know that $(1-\alpha)100\%$ of the z-distribution lies in the interval $\pm z_{\alpha/2}$. We can express this as

$$\Pr(-z_{\alpha/2} < z < z_{\alpha/2}) = 1-\alpha. \tag{6.A2.1}$$

Equation 6.A2.1 should be read as follows: *The probability that a randomly chosen z-score will fall in the interval $\pm z_{\alpha/2}$ is $1-\alpha$.*

Here, we will concentrate on the inequality within the colored parentheses in equation 6.A2.1 ($-z_{\alpha/2} < z < z_{\alpha/2}$), and we will show that it can be rearranged into a more familiar form without changing its meaning. The colored text will be retained to remind us that the following transformations do not change the meaning of what is inside the parentheses.

If we insert the definition of the z-score into equation 6.A2.1, we obtain the following:

$$\Pr\left(-z_{\alpha/2} < \frac{m-\mu}{\sigma_m} < z_{\alpha/2}\right) = 1-\alpha. \tag{6.A2.2}$$

Multiplying all three terms by σ_m leaves us with this:

$$\Pr(-z_{\alpha/2}(\sigma_m) < m-\mu < z_{\alpha/2}(\sigma_m)) = 1-\alpha. \tag{6.A2.3}$$

If we then subtract m from all three terms, we are left with the following:

$$\Pr(-z_{\alpha/2}(\sigma_m) - m < -\mu < z_{\alpha/2}(\sigma_m) - m) = 1-\alpha. \tag{6.A2.4}$$

This transformation leaves $-\mu$ in the center of the inequality. If we now multiply all three terms by -1 (to make μ positive) and reorder the terms, we obtain

$$\Pr(m + z_{\alpha/2}(\sigma_m) > \mu > m - z_{\alpha/2}(\sigma_m)) = 1-\alpha. \tag{6.A2.5}$$

Because we multiplied by -1, the directions of the inequality signs changed. For example, if we multiply all three numbers in this inequality $1 < 2 < 3$ by -1, then we would obtain $-1 > -2 > -3$. Equation 6.A2.5 shows that μ is between $m + z_{\alpha/2}(\sigma_m)$ and $m - z_{\alpha/2}(\sigma_m)$ $(1-\alpha)100\%$ of the time. In other words, the result we've obtained says that the probability is $1-\alpha$ that the interval

$$m \pm z_{\alpha/2}(\sigma_m)$$

will contain μ. We can state this as follows:

$$\Pr(\mu \text{ is in } m \pm z_{\alpha/2}(\sigma_m)) = 1-\alpha.$$

This means that for $(1-\alpha)100\%$ of all sample means, the interval $m \pm z_{\alpha/2}(\sigma_m)$ will capture μ no matter what μ and σ are or what n is. This is a really nice result.

APPENDIX 6.3: PRECISION PLANNING AND THE MARGIN OF ERROR

Throughout Chapter 6, we've noted that the width of a confidence interval depends on sample size. All things being equal (i.e., σ and α fixed), the width of a confidence interval decreases as sample size increases. Because sample size is chosen by the researcher, we have considerable control over the precision of the estimates that we make. Of course, we could make very precise parameter estimates if our sample sizes were in the thousands. However, practical considerations always enter into research, and it would be expensive and time-consuming to obtain extremely large sample sizes for each estimate we wish to make. For example, we may need to pay research assistants to contact participants and make the measurements that interest us. In addition, we may have to spend a long time administering tests to each member of our sample, as in the case of measuring IQ. Therefore, before we make any measurements, we should ask ourselves how precise we would like our measurements to be and then determine the sample size required to achieve the desired level of precision.

Margin of Error

The definition of a confidence interval for a mean is often expressed as $m \pm moe$, where *moe* stands for *margin of error*. When σ is known, the margin of error for a sample mean is $z_{\alpha/2}(\sigma_m)$. Therefore, the *moe* depends on σ, α, and *n*. If we choose to compute the 95% confidence interval when σ is known, the only thing we have available to adjust *moe* is *n*.

In which units should we express our desired precision for the *moe*? A straightforward approach is to express precision in terms of σ. For example, we might wish the *moe* to be $f * \sigma$, where *f* could be any number greater than 0. Although *f can* be any number greater than 0, we will see that only numbers less than 1 are useful and those too close to 0 are impractical.

Let's state formally that

$$moe = z_{\alpha/2}\left(\sigma_m\right) = z_{\alpha/2}\left(\frac{\sigma}{\sqrt{n}}\right) = \frac{z_{\alpha/2}\,(\sigma)}{\sqrt{n}}. \qquad (6.A3.1)$$

If we would like *moe* to be $f * \sigma$, then we can rewrite equation 6.A3.1 as follows:

$$f * \sigma = \frac{z_{\alpha/2}\,(\sigma)}{\sqrt{n}}. \qquad (6.A3.2)$$

A little manipulation of equation 6.A3.2 will show us what sample size we need to achieve our desired *moe*. Multiplying both sides of equation 6.A3.2 by $\sqrt{n}$ moves it over to the left.

$$\sqrt{n} * f * \sigma = z_{\alpha/2}(\sigma). \qquad (6.A3.3)$$

Because σ appears on both sides of equation 6.A3.3, we can simply divide both sides by σ to get

$$\sqrt{n} * f = z_{\alpha/2}. \qquad (6.A3.4)$$

This is really interesting because σ has disappeared from our definition of *f*. When we now divide both sides of equation 6.A3.4 by *f*, we find that $\sqrt{n}$ is completely defined in terms of $z_{\alpha/2}$ and *f*:

$$\sqrt{n} = z_{\alpha/2}/f. \qquad (6.A3.5)$$

Squaring both sides will give us what we want:

$$n = (z_{\alpha/2}/f)^2. \qquad (6.A3.6)$$

Equation 6.A3.6 shows that it is very simple to determine the sample size required to achieve an *moe* that is expressed as some fraction (*f*) of σ.

If we consider a few examples, we'll be able to see how this works. For all of these examples, we'll let α = .05 so that $z_{\alpha/2} = 1.96$. Let's say we want *moe* = σ. In this case, *f* = 1. Therefore, equation 6.A3.6 tells us that $n = (1.96/1)^2 = 1.96^2 = 3.84$ (rounded to two decimal places). Of course, it's not possible to have a sample of 3.84 individuals, so we will always round to the next integer (i.e., 4 in this case). Therefore, to achieve an *moe* equal to the population standard deviation, we would need 4 scores in our sample. You can imagine that a sample of 4 scores would produce a rather imprecise estimate of μ. Therefore, values of *f* greater than or equal to 1 are not useful.

Let's say we want *moe* = σ/4. In this case, *f* = .25. Therefore, equation 6.A3.6 tells us that $n = (1.96/.25)^2 = 7.84^2 = 61.46$ (rounded to two decimal places). Once again, we round up to the next integer to get a sample size of *n* = 62. Therefore, to achieve an *moe* equal to one-quarter of the population standard deviation, we would need 62 scores in our sample.

As a final example, let's say we want *moe* = σ/10. In this case, *f* = .1. Therefore, equation 6.A3.6 tells us that $n = (1.96/.1)^2 = 19.6^2 = 384.16$ (rounded to two decimal places). Rounding up to the next integer gives a sample size of *n* = 385. Therefore, to achieve an *moe* equal to one-tenth of the population standard deviation, we would need 385 scores in our sample.

As *f* approaches 0, the sample size required to achieve an *moe* equal to $f * \sigma$ increases very quickly. You can verify this by finding the sample size required to obtain an *moe* equal to $.001 * \sigma$. This is why values of *f* close to 0 are impractical.

Planning a Study

When we plan a study, we have competing needs. On the one hand, we want to have the most precise measure possible; on the other hand, greater precision generally entails increased cost. If an extremely precise measure is required but obtaining it is prohibitively expensive, it would make no sense to try to make do with a smaller sample that would produce less precision than required.

To make these concepts more concrete, we'll consider a situation in which money is at stake.

First, imagine that the Prescott Pharmaceutical Company (PPC) is working on a drug to improve attentional focus (AF) in adolescents with attention deficit disorder (ADD). Because they don't have the resources to conduct this research in-house, PPC offers you a contract to do this research. They require an estimate, with 95% confidence, of the mean AF in the population of all adolescents. They also require the *moe* to be $\sigma/10$. (Although they haven't told you why they need such a precise measurement, we can assume that they have good reasons.) Because PPC has contracted many similar studies in the past, they can tell prospective researchers that $\sigma = 25$. PPC offers you a contract of $50,000 to do this research. You consider this offer because you have statistics training. With a little research of your own, you find that it will cost you $95.00 to obtain an AF measure from each participant in your study. The $95.00 covers all costs involved, including lab work, research assistant pay, transportation, and so forth. Would it be worth your while to accept this contract?

To answer this question, you would need to know whether it would cost you more than $50,000 to get the measurement required by PPC. Therefore, you must determine the number of participants needed to achieve an *moe* of $\sigma/10$ for a 95% confidence interval. Given your expertise, you know that the number of participants required is $n = (z_{\alpha/2}/f)^2$. Because you know that $z_{\alpha/2} = 1.96$ and $f = .1$, you are able to determine that you would need 385 participants to achieve the required *moe* of $\sigma/10$. Therefore, your cost would be 385 * $95 = $36,575. Your profit would be $50,000 – $36,575 = $13,425. That's not bad, so you might consider accepting this contract.