Chapter 10: Linear Systems of Equations and Eigenvalues

1. (a) We can rewrite this system of equations in terms of matrices:

$$\begin{bmatrix} -3 & 5 & 5 \\ 1 & -4 & -2 \\ 3 & 0 & -4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -43 \\ 31 \\ 7 \end{bmatrix}.$$

We can solve this system by left-multiplying both sides of this equation by the inverse of the (3×3) matrix. To find this matrix, we first determine the minor elements:

$$M_{11} = \left| \begin{bmatrix} -4 & -2 \\ 0 & -4 \end{bmatrix} \right| = (-4 \times -4) - (-2 \times 0) = 16$$

$$M_{12} = \left| \begin{bmatrix} 1 & -2 \\ 3 & -4 \end{bmatrix} \right| = (1 \times -4) - (-2 \times 3) = 2$$

$$M_{13} = \left| \begin{bmatrix} 1 & -4 \\ 3 & 0 \end{bmatrix} \right| = (1 \times 0) - (-4 \times 3) = 12$$

$$M_{21} = \left| \begin{bmatrix} 5 & 5 \\ 0 & -4 \end{bmatrix} \right| = (5 \times -4) - (5 \times 0) = -20$$

$$M_{22} = \left| \begin{bmatrix} -3 & 5 \\ 3 & -4 \end{bmatrix} \right| = (-3 \times -4) - (5 \times 3) = -3$$

$$M_{23} = \left| \begin{bmatrix} -3 & 5 \\ 3 & 0 \end{bmatrix} \right| = (-3 \times 0) - (5 \times 3) = -15$$

$$M_{31} = \left| \begin{bmatrix} 5 & 5 \\ -4 & -2 \end{bmatrix} \right| = (5 \times -2) - (5 \times -4) = 10$$

$$M_{32} = \left| \begin{bmatrix} -3 & 5 \\ 1 & -2 \end{bmatrix} \right| = (-3 \times -2) - (5 \times 1) = 1$$

$$M_{33} = \left| \begin{bmatrix} -3 & 5 \\ 1 & -4 \end{bmatrix} \right| = (-3 \times -4) - (5 \times 1) = 7$$

Next we find the cofactors,

$$C_{11} = -1^{(1+1)} M_{11} = 16, \quad C_{12} = -1^{(1+2)} M_{12} = -2, \quad C_{13} = -1^{(1+3)} M_{13} = 12,$$

$$C_{21} = -1^{(2+1)} M_{21} = 20, \quad C_{22} = -1^{(2+2)} M_{22} = -3, \quad C_{23} = -1^{(2+3)} M_{23} = 15,$$

$$C_{31} = -1^{(3+1)} M_{31} = 10, \quad C_{32} = -1^{(3+2)} M_{32} = -1, \quad C_{33} = -1^{(3+3)} M_{33} = 7,$$

and take the transpose of the cofactor matrix to find the adjoint matrix,

$$\mathrm{adj}\left(\begin{bmatrix} -3 & 5 & 5 \\ 1 & -4 & -2 \\ 3 & 0 & -4 \end{bmatrix} \right) = \begin{bmatrix} 16 & 20 & 10 \\ -2 & -3 & -1 \\ 12 & 15 & 7 \end{bmatrix}.$$

To find the determinant, we choose one row or column (the first row in this case), multiply the elements by their cofactors, and add the products:

$$\left| \begin{bmatrix} -3 & 5 & 5 \\ 1 & -4 & -2 \\ 3 & 0 & -4 \end{bmatrix} \right| = (-3 \times 16) + (5 \times -2) + (5 \times 12) = 2.$$

We then plug the determinant and the adjoint matrix into the formula for a matrix inverse:

$$\left(\begin{bmatrix} 1 & 3 & 1 \\ 2 & 1 & 5 \\ 2 & 2 & -3 \end{bmatrix}\right)^{-1} = \frac{1}{2}\begin{bmatrix} 16 & 20 & 10 \\ -2 & -3 & -1 \\ 12 & 15 & 7 \end{bmatrix} = \begin{bmatrix} 8 & 10 & 5 \\ -1 & -1.5 & -0.5 \\ 6 & 7.5 & 3.5 \end{bmatrix}.$$

The solution to the system of equations is therefore

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 8 & 10 & 5 \\ -1 & -1.5 & -0.5 \\ 6 & 7.5 & 3.5 \end{bmatrix}\begin{bmatrix} -43 \\ 31 \\ 7 \end{bmatrix} = \begin{bmatrix} 1 \\ -7 \\ -1 \end{bmatrix}.$$

(b) We can rewrite this system of equations in terms of matrices:

$$\begin{bmatrix} -2 & 3 & 0 \\ -4 & 1 & 3 \\ 0 & 5 & -5 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ -5 \\ -15 \end{bmatrix}.$$

We can solve this system by left-multiplying both sides of this equation by the inverse of the (3×3) matrix. To find this matrix, we first determine the minor elements:

$$\begin{aligned}
M_{11} &= \left|\begin{bmatrix} 1 & 3 \\ 5 & -5 \end{bmatrix}\right| &&= (1 \times -5) - (3 \times 5) &&= -20 \\
M_{12} &= \left|\begin{bmatrix} -4 & 3 \\ 0 & -5 \end{bmatrix}\right| &&= (-4 \times -5) - (3 \times 0) &&= 20 \\
M_{13} &= \left|\begin{bmatrix} -4 & 1 \\ 0 & 5 \end{bmatrix}\right| &&= (-4 \times 5) - (1 \times 0) &&= -20 \\
M_{21} &= \left|\begin{bmatrix} 3 & 0 \\ 5 & -5 \end{bmatrix}\right| &&= (3 \times -5) - (0 \times 5) &&= -15 \\
M_{22} &= \left|\begin{bmatrix} -2 & 0 \\ 0 & -5 \end{bmatrix}\right| &&= (-2 \times -5) - (0 \times 0) &&= 10 \\
M_{23} &= \left|\begin{bmatrix} -2 & 3 \\ 0 & 5 \end{bmatrix}\right| &&= (-2 \times 5) - (3 \times 0) &&= -10 \\
M_{31} &= \left|\begin{bmatrix} 3 & 0 \\ 1 & 3 \end{bmatrix}\right| &&= (3 \times 3) - (0 \times 1) &&= 9 \\
M_{32} &= \left|\begin{bmatrix} -2 & 0 \\ -4 & 3 \end{bmatrix}\right| &&= (-2 \times 3) - (0 \times -4) &&= -6 \\
M_{33} &= \left|\begin{bmatrix} -2 & 3 \\ -4 & 1 \end{bmatrix}\right| &&= (-2 \times 1) - (3 \times -4) &&= 10
\end{aligned}$$

Next we find the cofactors,

$$\begin{aligned}
&C_{11} = -1^{(1+1)}M_{11} = -20, \quad C_{12} = -1^{(1+2)}M_{12} = -20, \quad C_{13} = -1^{(1+3)}M_{13} = -20, \\
&C_{21} = -1^{(2+1)}M_{21} = 15, \quad C_{22} = -1^{(2+2)}M_{22} = 10, \quad C_{23} = -1^{(2+3)}M_{23} = 10, \\
&C_{31} = -1^{(3+1)}M_{31} = 9, \quad C_{32} = -1^{(3+2)}M_{32} = 6, \quad C_{33} = -1^{(3+3)}M_{33} = 10,
\end{aligned}$$

and take the transpose of the cofactor matrix to find the adjoint matrix,

$$\operatorname{adj}\left(\begin{bmatrix} -2 & 3 & 0 \\ -4 & 1 & 3 \\ 0 & 5 & -5 \end{bmatrix}\right) = \begin{bmatrix} -20 & 15 & 9 \\ -20 & 10 & 6 \\ -20 & 10 & 10 \end{bmatrix}.$$

To find the determinant, we choose one row or column (the first row in this case), multiply the elements by their cofactors, and add the products:

$$\left|\begin{bmatrix} -2 & 3 & 0 \\ -4 & 1 & 3 \\ 0 & 5 & -5 \end{bmatrix}\right| = (-2 \times -20) + (3 \times -20) + (0 \times -20) = -20.$$

We then plug the determinant and the adjoint matrix into the formula for a matrix inverse:

$$\left(\begin{bmatrix} 1 & 3 & -4 \\ 2 & 1 & 3 \\ 2 & 2 & -2 \end{bmatrix}\right)^{-1} = \frac{1}{-20}\begin{bmatrix} -20 & 15 & 9 \\ -20 & 10 & 6 \\ -20 & 10 & 10 \end{bmatrix} = \begin{bmatrix} 1 & -0.75 & -0.45 \\ 1 & -0.5 & -0.3 \\ 1 & -0.5 & -0.5 \end{bmatrix}.$$

The solution to the system of equations is therefore

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & -0.75 & -0.45 \\ 1 & -0.5 & -0.3 \\ 1 & -0.5 & -0.5 \end{bmatrix}\begin{bmatrix} 5 \\ -5 \\ -15 \end{bmatrix} = \begin{bmatrix} 15.5 \\ 12 \\ 15 \end{bmatrix}.$$

(c) We can rewrite this system of equations in terms of matrices:

$$\begin{bmatrix} -2 & 2 & -1 \\ 5 & -3 & -4 \\ 3 & 0 & -3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ -11 \\ -12 \end{bmatrix}.$$

We can solve this system by left-multiplying both sides of this equation by the inverse of the (3×3)

matrix. To find this matrix, we first determine the minor elements:

$$
\begin{aligned}
M_{11} &= \left|\begin{bmatrix} -3 & -4 \\ 0 & -3 \end{bmatrix}\right| &&= (-3 \times -3) - (-4 \times 0) &&= 9 \\
M_{12} &= \left|\begin{bmatrix} 5 & -4 \\ 3 & -3 \end{bmatrix}\right| &&= (5 \times -3) - (-4 \times 3) &&= -3 \\
M_{13} &= \left|\begin{bmatrix} 5 & -3 \\ 3 & 0 \end{bmatrix}\right| &&= (5 \times 0) - (-3 \times 3) &&= 9 \\
M_{21} &= \left|\begin{bmatrix} 2 & -1 \\ 0 & -3 \end{bmatrix}\right| &&= (2 \times -3) - (-1 \times 0) &&= -6 \\
M_{22} &= \left|\begin{bmatrix} -2 & -1 \\ 3 & -3 \end{bmatrix}\right| &&= (-2 \times -3) - (-1 \times 3) &&= 9 \\
M_{23} &= \left|\begin{bmatrix} -2 & 2 \\ 3 & 0 \end{bmatrix}\right| &&= (-2 \times 0) - (2 \times 3) &&= -6 \\
M_{31} &= \left|\begin{bmatrix} 2 & -1 \\ -3 & -4 \end{bmatrix}\right| &&= (2 \times -4) - (-1 \times -3) &&= -11 \\
M_{32} &= \left|\begin{bmatrix} -2 & -1 \\ 5 & -4 \end{bmatrix}\right| &&= (-2 \times -4) - (-1 \times 5) &&= 13 \\
M_{33} &= \left|\begin{bmatrix} -2 & 2 \\ 5 & -3 \end{bmatrix}\right| &&= (-2 \times -3) - (2 \times 5) &&= -4
\end{aligned}
$$

Next we find the cofactors,

$$
\begin{aligned}
&C_{11} = -1^{(1+1)} M_{11} = 9, \quad && C_{12} = -1^{(1+2)} M_{12} = 3, \quad && C_{13} = -1^{(1+3)} M_{13} = 9, \\
&C_{21} = -1^{(2+1)} M_{21} = 6, && C_{22} = -1^{(2+2)} M_{22} = 9, && C_{23} = -1^{(2+3)} M_{23} = 6, \\
&C_{31} = -1^{(3+1)} M_{31} = -11, && C_{32} = -1^{(3+2)} M_{32} = -13, && C_{33} = -1^{(3+3)} M_{33} = -4,
\end{aligned}
$$

and take the transpose of the cofactor matrix to find the adjoint matrix,

$$
\operatorname{adj}\left(\begin{bmatrix} -2 & 2 & -1 \\ 5 & -3 & -4 \\ 3 & 0 & -3 \end{bmatrix}\right) = \begin{bmatrix} 9 & 6 & -11 \\ 3 & 9 & -13 \\ 9 & 6 & -4 \end{bmatrix}.
$$

To find the determinant, we choose one row or column (the first row in this case), multiply the elements by their cofactors, and add the products:

$$
\left|\begin{bmatrix} -2 & 2 & -1 \\ 5 & -3 & -4 \\ 3 & 0 & -3 \end{bmatrix}\right| = (-2 \times 9) + (2 \times 3) + (-1 \times 9) = -21.
$$

We then plug the determinant and the adjoint matrix into the formula for a matrix inverse:

$$
\left(\begin{bmatrix} 1 & 3 & 5 \\ 2 & 1 & 2 \\ 2 & 2 & -2 \end{bmatrix}\right)^{-1} = \frac{1}{-21}\begin{bmatrix} 9 & 6 & -11 \\ 3 & 9 & -13 \\ 9 & 6 & -4 \end{bmatrix} = \begin{bmatrix} -0.43 & -0.29 & 0.52 \\ -0.14 & -0.43 & 0.62 \\ -0.43 & -0.29 & 0.19 \end{bmatrix}.
$$

The solution to the system of equations is therefore

$$
\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -0.43 & -0.29 & 0.52 \\ -0.14 & -0.43 & 0.62 \\ -0.43 & -0.29 & 0.19 \end{bmatrix}\begin{bmatrix} 2 \\ -11 \\ -12 \end{bmatrix} = \begin{bmatrix} -4 \\ -3 \\ 0 \end{bmatrix}.
$$

(d) We can rewrite this system of equations in terms of matrices:

$$\begin{bmatrix} 2 & -1 & 1 \\ -2 & 4 & 5 \\ -3 & 3 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ -1 \\ -2 \end{bmatrix}.$$

We can solve this system by left-multiplying both sides of this equation by the inverse of the (3×3) matrix. To find this matrix, we first determine the minor elements:

$$\begin{aligned}
M_{11} &= \left|\begin{bmatrix} 4 & 5 \\ 3 & -1 \end{bmatrix}\right| &= (4 \times -1) - (5 \times 3) &= -19 \\
M_{12} &= \left|\begin{bmatrix} -2 & 5 \\ -3 & -1 \end{bmatrix}\right| &= (-2 \times -1) - (5 \times -3) &= 17 \\
M_{13} &= \left|\begin{bmatrix} -2 & 4 \\ -3 & 3 \end{bmatrix}\right| &= (-2 \times 3) - (4 \times -3) &= 6 \\
M_{21} &= \left|\begin{bmatrix} -1 & 1 \\ 3 & -1 \end{bmatrix}\right| &= (-1 \times -1) - (1 \times 3) &= -2 \\
M_{22} &= \left|\begin{bmatrix} 2 & 1 \\ -3 & -1 \end{bmatrix}\right| &= (2 \times -1) - (1 \times -3) &= 1 \\
M_{23} &= \left|\begin{bmatrix} 2 & -1 \\ -3 & 3 \end{bmatrix}\right| &= (2 \times 3) - (-1 \times -3) &= 3 \\
M_{31} &= \left|\begin{bmatrix} -1 & 1 \\ 4 & 5 \end{bmatrix}\right| &= (-1 \times 5) - (1 \times 4) &= -9 \\
M_{32} &= \left|\begin{bmatrix} 2 & 1 \\ -2 & 5 \end{bmatrix}\right| &= (2 \times 5) - (1 \times -2) &= 12 \\
M_{33} &= \left|\begin{bmatrix} 2 & -1 \\ -2 & 4 \end{bmatrix}\right| &= (2 \times 4) - (-1 \times -2) &= 6
\end{aligned}$$

Next we find the cofactors,

$$\begin{aligned}
&C_{11} = -1^{(1+1)}M_{11} = -19, \quad C_{12} = -1^{(1+2)}M_{12} = -17, \quad C_{13} = -1^{(1+3)}M_{13} = 6, \\
&C_{21} = -1^{(2+1)}M_{21} = 2, \quad C_{22} = -1^{(2+2)}M_{22} = 1, \quad C_{23} = -1^{(2+3)}M_{23} = -3, \\
&C_{31} = -1^{(3+1)}M_{31} = -9, \quad C_{32} = -1^{(3+2)}M_{32} = -12, \quad C_{33} = -1^{(3+3)}M_{33} = 6,
\end{aligned}$$

and take the transpose of the cofactor matrix to find the adjoint matrix,

$$\mathrm{adj}\left(\begin{bmatrix} 2 & -1 & 1 \\ -2 & 4 & 5 \\ -3 & 3 & -1 \end{bmatrix}\right) = \begin{bmatrix} -19 & 2 & -9 \\ -17 & 1 & -12 \\ 6 & -3 & 6 \end{bmatrix}.$$

To find the determinant, we choose one row or column (the first row in this case), multiply the elements by their cofactors, and add the products:

$$\left|\begin{bmatrix} 2 & -1 & 1 \\ -2 & 4 & 5 \\ -3 & 3 & -1 \end{bmatrix}\right| = (2 \times -19) + (-1 \times -17) + (1 \times 6) = -15.$$

We then plug the determinant and the adjoint matrix into the formula for a matrix inverse:

$$\left(\begin{bmatrix} 1 & 3 & -2 \\ 2 & 1 & -1 \\ 2 & 2 & 2 \end{bmatrix}\right)^{-1} = \frac{1}{-15}\begin{bmatrix} -19 & 2 & -9 \\ -17 & 1 & -12 \\ 6 & -3 & 6 \end{bmatrix} = \begin{bmatrix} 1.27 & -0.13 & 0.6 \\ 1.13 & -0.07 & 0.8 \\ -0.4 & 0.2 & -0.4 \end{bmatrix}.$$

The solution to the system of equations is therefore

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1.27 & -0.13 & 0.6 \\ 1.13 & -0.07 & 0.8 \\ -0.4 & 0.2 & -0.4 \end{bmatrix} \begin{bmatrix} 4 \\ -1 \\ -2 \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \\ -1 \end{bmatrix}.$$

2. (a) The augmented matrix that corresponds to this system is

$$\left[\begin{array}{rrr|r} -5 & 5 & 6 & 2 \\ -3 & 3 & -4 & -14 \\ 9 & 6 & 5 & -17 \end{array}\right].$$

We break the first three columns down to an identity matrix using the elementary row operations. Each time we perform an operation, we also apply it to the last column. Once we've created an identity matrix in the first three columns, then the values of the elements in the last column are the (x, y, z) solution to the system.

First we multiply the first row by 3 (to make it easier to eliminate the first element), and interchange the rows to that the first is third, the second is first, and the third is second,

$$\left[\begin{array}{rrr|r} -3 & 3 & -4 & -14 \\ 9 & 6 & 5 & -17 \\ -15 & 15 & 18 & 6 \end{array}\right].$$

Next we multiply the first row by 3 and add it to the second row, and we multiply the first row by -5 and add it to the third row,

$$\left[\begin{array}{rrr|r} -3 & 3 & -4 & -14 \\ 0 & 15 & -7 & -59 \\ 0 & 0 & 38 & 76 \end{array}\right].$$

We divide row 3 by 2,

$$\left[\begin{array}{rrr|r} -3 & 3 & -4 & -14 \\ 0 & 15 & -7 & -59 \\ 0 & 0 & 1 & 2 \end{array}\right],$$

and add 7 times the third row to the second row, and 4 times the third row to the first row,

$$\left[\begin{array}{rrr|r} -3 & 3 & 0 & -6 \\ 0 & 15 & 0 & -45 \\ 0 & 0 & 1 & 2 \end{array}\right].$$

Next we divide the second row by 15,

$$\left[\begin{array}{rrr|r} -3 & 3 & 0 & -6 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 2 \end{array}\right],$$

and multiply it by -3 and add it to the first row,

$$\left[\begin{array}{rrr|r} -3 & 0 & 0 & 3 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 2 \end{array}\right].$$

Finally we the first row by -3,

$$\left[\begin{array}{rrr|r} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 2 \end{array}\right].$$

The reduction implies that the system's solution is

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 \\ -3 \\ 2 \end{bmatrix}.$$

(b) The augmented matrix that corresponds to this system is

$$\left[\begin{array}{ccc|c} -6 & -4 & 1 & 10 \\ 1 & 4 & -4 & 16 \\ -3 & 5 & 4 & 19 \end{array}\right].$$

We break the first three columns down to an identity matrix using the elementary row operations. Each time we perform an operation, we also apply it to the last column. Once we've created an identity matrix in the first three columns, then the values of the elements in the last column are the (x, y, z) solution to the system.

First we interchange the rows,

$$\left[\begin{array}{ccc|c} 1 & 4 & -4 & 16 \\ -3 & 5 & 4 & 19 \\ -6 & -4 & 1 & 10 \end{array}\right],$$

then add 3 times the first row to the second row, and 6 times the first row to the third,

$$\left[\begin{array}{ccc|c} 1 & 4 & -4 & 16 \\ 0 & 17 & -8 & 67 \\ 0 & 20 & -23 & 106 \end{array}\right].$$

This next part is very tricky. 17 and 20 share no factors, so to avoid fractions we multiply the second row by 20 and the third row by 17,

$$\left[\begin{array}{ccc|c} 1 & 4 & -4 & 16 \\ 0 & 340 & -160 & 1340 \\ 0 & 340 & -391 & 1802 \end{array}\right],$$

we add -1 times the second row to the third,

$$\left[\begin{array}{ccc|c} 1 & 4 & -4 & 16 \\ 0 & 340 & -160 & 1340 \\ 0 & 0 & -231 & 462 \end{array}\right],$$

and we divide the last row by -231,

$$\left[\begin{array}{ccc|c} 1 & 4 & -4 & 16 \\ 0 & 340 & -160 & 1340 \\ 0 & 0 & 1 & -2 \end{array}\right].$$

Next we multiply the third row by 160 and add it to the second row,

$$\left[\begin{array}{ccc|c} 1 & 4 & -4 & 16 \\ 0 & 340 & 0 & 1020 \\ 0 & 0 & 1 & -2 \end{array}\right],$$

and we multiply the third row by 4 and add it to the first row,

$$\left[\begin{array}{ccc|c} 1 & 4 & 0 & 8 \\ 0 & 340 & 0 & 1020 \\ 0 & 0 & 1 & -2 \end{array}\right].$$

We divide the second row by 340,

$$\left[\begin{array}{ccc|c} 1 & 4 & 0 & 8 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -2 \end{array}\right],$$

and multiply it by -4 and add it to the first row,

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & -4 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -2 \end{array}\right].$$

The reduction implies that the system's solution is

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -4 \\ 3 \\ -2 \end{bmatrix}.$$

(c) The augmented matrix that corresponds to this system is

$$\left[\begin{array}{ccc|c} 5 & -1 & -5 & -3 \\ -5 & -5 & 6 & -18 \\ 0 & -2 & -6 & 12 \end{array}\right].$$

We break the first three columns down to an identity matrix using the elementary row operations. Each time we perform an operation, we also apply it to the last column. Once we've created an identity matrix in the first three columns, then the values of the elements in the last column are the (x, y, z) solution to the system.

We start by adding the first row to the second,

$$\left[\begin{array}{ccc|c} 5 & -1 & -5 & -3 \\ 0 & -6 & 1 & -21 \\ 0 & -2 & -6 & 12 \end{array}\right],$$

interchanging the second and third rows,

$$\left[\begin{array}{ccc|c} 5 & -1 & -5 & -3 \\ 0 & -2 & -6 & 12 \\ 0 & -6 & 1 & -21 \end{array}\right],$$

dividing the second row by -2,

$$\left[\begin{array}{ccc|c} 5 & -1 & -5 & -3 \\ 0 & 1 & 3 & -6 \\ 0 & -6 & 1 & -21 \end{array}\right],$$

and adding 6 times the second row to the third,

$$\left[\begin{array}{ccc|c} 5 & -1 & -5 & -3 \\ 0 & 1 & 3 & -6 \\ 0 & 0 & 19 & -57 \end{array}\right].$$

Next we divide the third row by 19,

$$\left[\begin{array}{ccc|c} 5 & -1 & -5 & -3 \\ 0 & 1 & 3 & -6 \\ 0 & 0 & 1 & -3 \end{array}\right],$$

add -3 times the third row to the second,

$$\left[\begin{array}{rrr|r} 5 & -1 & -5 & -3 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -3 \end{array}\right],$$

and add 5 times the third row to the first,

$$\left[\begin{array}{rrr|r} 5 & -1 & 0 & -18 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -3 \end{array}\right].$$

Finally we add the second row to the first,

$$\left[\begin{array}{rrr|r} 5 & 0 & 0 & -15 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -3 \end{array}\right],$$

and divide the first row by 5,

$$\left[\begin{array}{rrr|r} 1 & 0 & 0 & -3 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -3 \end{array}\right].$$

The reduction implies that the system's solution is

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -3 \\ 3 \\ -3 \end{bmatrix}.$$

(d) The augmented matrix that corresponds to this system is

$$\left[\begin{array}{rrrr|r} -1 & 3 & -6 & -3 & 0 \\ 1 & -1 & 1 & 4 & -11 \\ 5 & -5 & -1 & -2 & -1 \\ 2 & 3 & -3 & 0 & -3 \end{array}\right].$$

We break the first four columns down to an identity matrix using the elementary row operations. Each time we perform an operation, we also apply it to the last column. Once we've created an identity matrix in the first three columns, then the values of the elements in the last column are the (w, x, y, z) solution to the system.

We start by interchanging the first two rows,

$$\left[\begin{array}{rrrr|r} 1 & -1 & 1 & 4 & -11 \\ -1 & 3 & -6 & -3 & 0 \\ 5 & -5 & -1 & -2 & -1 \\ 2 & 3 & -3 & 0 & -3 \end{array}\right],$$

and adding the first row to the second, -5 times the first row to the third, and -2 times the first row to the fourth,

$$\left[\begin{array}{rrrr|r} 1 & -1 & 1 & 4 & -11 \\ 0 & 2 & -5 & 1 & -11 \\ 0 & 0 & -6 & -22 & 54 \\ 0 & 5 & -5 & -8 & 19 \end{array}\right].$$

Next we multiply the fourth row by 2,

$$\left[\begin{array}{rrrr|r} 1 & -1 & 1 & 4 & -11 \\ 0 & 2 & -5 & 1 & -11 \\ 0 & 0 & -6 & -22 & 54 \\ 0 & 10 & -10 & -16 & 38 \end{array}\right],$$

and add -5 times the second row to the fourth row,

$$\left[\begin{array}{rrrr|r} 1 & -1 & 1 & 4 & -11 \\ 0 & 2 & -5 & 1 & -11 \\ 0 & 0 & -6 & -22 & 54 \\ 0 & 0 & 15 & -21 & 93 \end{array}\right].$$

We divide the third row by -2,

$$\left[\begin{array}{rrrr|r} 1 & -1 & 1 & 4 & -11 \\ 0 & 2 & -5 & 1 & -11 \\ 0 & 0 & 3 & 11 & -27 \\ 0 & 0 & 15 & -21 & 93 \end{array}\right],$$

add -5 times the third row to the fourth,

$$\left[\begin{array}{rrrr|r} 1 & -1 & 1 & 4 & -11 \\ 0 & 2 & -5 & 1 & -11 \\ 0 & 0 & 3 & 11 & -27 \\ 0 & 0 & 0 & -76 & 228 \end{array}\right],$$

and divide the fourth row by -76,

$$\left[\begin{array}{rrrr|r} 1 & -1 & 1 & 4 & -11 \\ 0 & 2 & -5 & 1 & -11 \\ 0 & 0 & 3 & 11 & -27 \\ 0 & 0 & 0 & 1 & -3 \end{array}\right].$$

Now we add -11 times the fourth row to the third row, -1 times the fourth row to the second row, and -4 times the fourth row to the first row,

$$\left[\begin{array}{rrrr|r} 1 & -1 & 1 & 0 & 1 \\ 0 & 2 & -5 & 0 & -8 \\ 0 & 0 & 3 & 0 & 6 \\ 0 & 0 & 0 & 1 & -3 \end{array}\right],$$

divide the third row by 3,

$$\left[\begin{array}{rrrr|r} 1 & -1 & 1 & 0 & 1 \\ 0 & 2 & -5 & 0 & -8 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & -3 \end{array}\right],$$

add 5 times the third row to the second row and -1 times the third row to the first row,

$$\left[\begin{array}{rrrr|r} 1 & -1 & 0 & 0 & -1 \\ 0 & 2 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & -3 \end{array}\right],$$

divide the second row by 2,

$$\left[\begin{array}{rrrr|r} 1 & -1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & -3 \end{array}\right],$$

and add the second row to the first,

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & -3 \end{array}\right].$$

The reduction implies that the system's solution is

$$\begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 2 \\ -3 \end{bmatrix}.$$

3. (a) The augmented matrix that corresponds to this system is

$$\left[\begin{array}{ccc|c} 1 & -2 & -1 & 15 \\ -1 & -1 & 1 & -6 \\ 1 & -6 & -1 & -43 \end{array}\right].$$

We start by adding the first row to the second, and adding -1 times the first row to the third,

$$\left[\begin{array}{ccc|c} 1 & -2 & -1 & 15 \\ 0 & -3 & 0 & 9 \\ 0 & -4 & 0 & -58 \end{array}\right].$$

We divide the second row by -3 and the third row by -4,

$$\left[\begin{array}{ccc|c} 1 & -2 & -1 & 15 \\ 0 & 1 & 0 & -3 \\ 0 & 1 & 0 & 14.5 \end{array}\right],$$

and add -2 times the second row to the first row, and -1 times the second row to the third row,

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & 21 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 0 & 20.5 \end{array}\right].$$

Remember that the rows of the augmented matrix refer to equations in the system, that the columns refer to the variables x, y, and z respectively, the elements are coefficients, and the vertical line is an equal sign. At this point we've reduced the system of equations to

$$\begin{cases} x - z = 21, \\ y = -3, \\ 0 = 20.5. \end{cases}$$

The third equation is an obviously untrue statement. Therefore this system of equations has no solution.

(b) The augmented matrix that corresponds to this system is

$$\left[\begin{array}{ccc|c} -4 & -1 & -2 & 15 \\ 1 & -2 & 2 & 18 \\ 0 & -6 & 4 & 20 \end{array}\right].$$

We start by moving the first row to the third position and moving the other two rows up,

$$\left[\begin{array}{rrr|r} 1 & -2 & 2 & 18 \\ 0 & -6 & 4 & 20 \\ -4 & -1 & -2 & 15 \end{array}\right].$$

Next we add 4 times the first row to the third row,

$$\left[\begin{array}{rrr|r} 1 & -2 & 2 & 18 \\ 0 & -6 & 4 & 20 \\ 0 & -9 & 6 & 87 \end{array}\right].$$

We divide the second row by 2,

$$\left[\begin{array}{rrr|r} 1 & -2 & 2 & 18 \\ 0 & -3 & 2 & 10 \\ 0 & -9 & 6 & 87 \end{array}\right],$$

then multiply it by -3 and add it to the third row,

$$\left[\begin{array}{rrr|r} 1 & -2 & 2 & 18 \\ 0 & -3 & 2 & 10 \\ 0 & 0 & 0 & 57 \end{array}\right].$$

The third equation is now 0=57, which is obviously untrue. Therefore the system has no solution.

(c) The augmented matrix that corresponds to this system is

$$\left[\begin{array}{rrr|r} -6 & -2 & 5 & -29 \\ 2 & -5 & 1 & -4 \\ 4 & 5 & -5 & 28 \end{array}\right].$$

Let's start by rearranging the rows,

$$\left[\begin{array}{rrr|r} 2 & -5 & 1 & -4 \\ 4 & 5 & -5 & 28 \\ -6 & -2 & 5 & -29 \end{array}\right],$$

and adding -2 times the first row to the second, and 3 times the first row to the third,

$$\left[\begin{array}{rrr|r} 2 & -5 & 1 & -4 \\ 0 & 15 & -7 & 36 \\ 0 & -17 & 8 & -41 \end{array}\right].$$

Next we multiply the second row by 17 and the third row by 15,

$$\left[\begin{array}{rrr|r} 2 & -5 & 1 & -4 \\ 0 & 255 & -119 & 612 \\ 0 & -255 & 120 & -615 \end{array}\right],$$

add the second row to the third,

$$\left[\begin{array}{rrr|r} 2 & -5 & 1 & -4 \\ 0 & 255 & -119 & 612 \\ 0 & 0 & 1 & -3 \end{array}\right],$$

then add 199 times the third row to the second,

$$\left[\begin{array}{rrr|r} 2 & -5 & 1 & -4 \\ 0 & 255 & 0 & 255 \\ 0 & 0 & 1 & -3 \end{array}\right],$$

and divide the second row by 255,

$$\left[\begin{array}{ccc|c} 2 & -5 & 1 & -4 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -3 \end{array}\right].$$

Finally, add -1 times the third row to the first row,

$$\left[\begin{array}{ccc|c} 2 & -5 & 0 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -3 \end{array}\right],$$

add 5 times the second row to the first row,

$$\left[\begin{array}{ccc|c} 2 & 0 & 0 & 4 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -3 \end{array}\right],$$

and divide the first row by 2,

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -3 \end{array}\right].$$

The reduction implies that the system has one unique solution, which is

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ -3 \end{bmatrix}.$$

(d) The augmented matrix that corresponds to this system is

$$\left[\begin{array}{ccc|c} -1 & 3 & -1 & 9 \\ 1 & -1 & 0 & -8 \\ -5 & 3 & 1 & 39 \end{array}\right].$$

We start by interchanging the first and second rows,

$$\left[\begin{array}{ccc|c} 1 & -1 & 0 & -8 \\ -1 & 3 & -1 & 9 \\ -5 & 3 & 1 & 39 \end{array}\right],$$

adding the first row to the second, and adding 5 times the first row to the third,

$$\left[\begin{array}{ccc|c} 1 & -1 & 0 & -8 \\ 0 & 2 & -1 & 1 \\ 0 & -2 & 1 & -1 \end{array}\right].$$

Next we add the second row to the third,

$$\left[\begin{array}{ccc|c} 1 & -1 & 0 & -8 \\ 0 & 2 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

The third equation in the system is now 0=0, which is an obviously true statement. Therefore this system has infinitely many solutions.

4. (a) The augmented matrix that corresponds to this system is

$$\left[\begin{array}{rrr|r} 1 & 3 & 1 & 11 \\ -1 & 5 & 2 & 4 \\ 4 & 4 & 1 & 29 \end{array}\right].$$

Let's start by adding the first row to the second, and adding -4 times the first row to the third,

$$\left[\begin{array}{rrr|r} 1 & 3 & 1 & 11 \\ 0 & 8 & 3 & 15 \\ 0 & -8 & -3 & -15 \end{array}\right].$$

Then we can add the second row to the third,

$$\left[\begin{array}{rrr|r} 1 & 3 & 1 & 11 \\ 0 & 8 & 3 & 15 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

We can see that this system has infinitely many solutions. Our task is now to derive a formula for these solutions, and to relate the specific solution. Let's continue reducing. We multiply the first row by 8,

$$\left[\begin{array}{rrr|r} 8 & 24 & 8 & 88 \\ 0 & 8 & 3 & 15 \\ 0 & 0 & 0 & 0 \end{array}\right],$$

and add -3 times the second row to the first row,

$$\left[\begin{array}{rrr|r} 8 & 0 & -1 & 43 \\ 0 & 8 & 3 & 15 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

Finally, we divide the first and second row by 8,

$$\left[\begin{array}{rrr|r} 1 & 0 & -\frac{1}{8} & \frac{43}{8} \\ 0 & 1 & \frac{3}{8} & \frac{15}{8} \\ 0 & 0 & 0 & 0 \end{array}\right].$$

The system of equations is now

$$\begin{cases} x - \frac{1}{8}z = \frac{43}{8}, \\ y + \frac{3}{8}z = \frac{15}{8}, \\ z = z, \end{cases}$$

where we add $z = z$ into the system as a general representation of the trivial statement 0=0. That is, by adding or subtracting the same thing to both sides of 0=0, we can turn this statement into $z = z$ for any value of z. Solving the system for x and y in terms of z gives us

$$\begin{cases} x = \frac{1}{8}z + \frac{43}{8}, \\ y = -\frac{3}{8}z + \frac{15}{8}, \\ z = z, \end{cases}$$

which can be written in matrix form as

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \frac{1}{8} \\ -\frac{3}{8} \\ 0 \end{bmatrix} + \begin{bmatrix} \frac{43}{8} \\ \frac{15}{8} \\ 1 \end{bmatrix} z.$$

z is a free variable, meaning that we can choose any value for z we want. But for any value of z there is only one solution to this system that can be found by plugging that value of z into the above equation. The specific solution is the name for the solution in which $z = 0$, which in this case is

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \frac{1}{8} \\ -\frac{3}{8} \\ 0 \end{bmatrix}.$$

(b) The augmented matrix that corresponds to this system is

$$\left[\begin{array}{rrr|r} 5 & 1 & 4 & -4 \\ -4 & 1 & -3 & -7 \\ -3 & 3 & -2 & -18 \end{array}\right].$$

Let's start by multiplying the second and third rows by 5,

$$\left[\begin{array}{rrr|r} 5 & 1 & 4 & -4 \\ -20 & 5 & -15 & -35 \\ -15 & 15 & -10 & -90 \end{array}\right].$$

Now we can add 4 times the first row to the second row and 3 times the first row to the third row,

$$\left[\begin{array}{rrr|r} 5 & 1 & 4 & -4 \\ 0 & 9 & 1 & -51 \\ 0 & 18 & 2 & -102 \end{array}\right],$$

and add -2 times the second row to the third row,

$$\left[\begin{array}{rrr|r} 5 & 1 & 4 & -4 \\ 0 & 9 & 1 & -51 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

The system has infinitely many solutions. Let's continue reducing by multiplying the first row by 9,

$$\left[\begin{array}{rrr|r} 45 & 9 & 36 & -36 \\ 0 & 9 & 1 & -51 \\ 0 & 0 & 0 & 0 \end{array}\right],$$

and adding -1 times the second row to the first,

$$\left[\begin{array}{rrr|r} 45 & 0 & 35 & 15 \\ 0 & 9 & 1 & -51 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

Finally, divide the first row by 45 and divide the second row by 9,

$$\left[\begin{array}{rrr|r} 1 & 0 & \frac{7}{9} & \frac{1}{3} \\ 0 & 1 & \frac{1}{9} & -\frac{51}{9} \\ 0 & 0 & 0 & 0 \end{array}\right].$$

The system of equations is now

$$\begin{cases} x + \frac{7}{9}z = \frac{1}{3}, \\ y + \frac{1}{9}z = -\frac{51}{9}, \\ z = z. \end{cases}$$

Solving the system for x and y in terms of z gives us

$$\begin{cases} x = \frac{1}{3} - \frac{7}{9}z, \\ \\ y = -\frac{51}{9} - \frac{1}{9}z, \\ \\ z = z, \end{cases}$$

which can be written in matrix form as

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \frac{1}{3} \\ \\ -\frac{51}{9} \\ \\ 0 \end{bmatrix} - \begin{bmatrix} \frac{7}{9} \\ \\ \frac{1}{9} \\ \\ 1 \end{bmatrix} z.$$

The specific solution in which $z = 0$ is

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \frac{1}{3} \\ \\ -\frac{51}{9} \\ \\ 0 \end{bmatrix}.$$

(c) The augmented matrix that corresponds to this system is

$$\left[\begin{array}{rrr|r} 1 & 6 & -1 & -18 \\ 3 & 2 & 1 & -22 \\ -5 & 6 & -4 & 18 \end{array}\right].$$

Let's start by adding -3 times the first row to the second, and 5 times the first row to the third,

$$\left[\begin{array}{rrr|r} 1 & 6 & -1 & -18 \\ 0 & -16 & 4 & 32 \\ 0 & 36 & -9 & -72 \end{array}\right].$$

We divide the second row by 4 and the third row by 9,

$$\left[\begin{array}{rrr|r} 1 & 6 & -1 & -18 \\ 0 & -4 & 1 & 8 \\ 0 & 4 & -1 & -8 \end{array}\right],$$

and add the second row to the third,

$$\left[\begin{array}{rrr|r} 1 & 6 & -1 & -18 \\ 0 & -4 & 1 & 8 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

This system has infinitely many solutions. To continue reducing, multiply the first row by 2,

$$\left[\begin{array}{rrr|r} 2 & 12 & -2 & -36 \\ 0 & -4 & 1 & 8 \\ 0 & 0 & 0 & 0 \end{array}\right],$$

add 3 times the second row to the first,

$$\left[\begin{array}{rrr|r} 2 & 0 & 1 & -4 \\ 0 & -4 & 1 & 8 \\ 0 & 0 & 0 & 0 \end{array}\right],$$

divide the first row by 2 and divide the second row by -4,

$$\left[\begin{array}{ccc|c} 1 & 0 & \frac{1}{2} & -2 \\ 0 & 1 & -\frac{1}{4} & -2 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

The system of equations is now

$$\begin{cases} x + \frac{1}{2}z = -2, \\ y - \frac{1}{4}z = -2, \\ z = z. \end{cases}$$

Solving the system for x and y in terms of z gives us

$$\begin{cases} x = -\frac{1}{2}z - 2, \\ y = -\frac{1}{4}z - 2, \\ z = z, \end{cases}$$

which can be written in matrix form as

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -2 \\ -2 \\ 0 \end{bmatrix} - \begin{bmatrix} \frac{1}{2} \\ \frac{1}{4} \\ 1 \end{bmatrix} z.$$

The specific solution in which $z = 0$ is

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -2 \\ -2 \\ 0 \end{bmatrix}.$$

(d) The augmented matrix that corresponds to this system is

$$\left[\begin{array}{ccc|c} 2 & -5 & -3 & 0 \\ 1 & -6 & -2 & 0 \\ 1 & 1 & -1 & 0 \end{array}\right].$$

Note that this system is a homogenous system, which means that it either has infinitely many solutions, or just the trivial (all zero) solution. Let's begin by rearranging the rows,

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 1 & -6 & -2 & 0 \\ 2 & -5 & -3 & 0 \end{array}\right],$$

then by adding -1 times the first row to the second, and -2 times the first row to the third,

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 0 & -7 & -1 & 0 \\ 0 & -7 & -1 & 0 \end{array}\right].$$

Now add -1 times the second row to the third,

$$\left[\begin{array}{rrr|r} 1 & 1 & -1 & 0 \\ 0 & -7 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

This system has infinitely many solutions. To continue reducing, multiply the top row by 7,

$$\left[\begin{array}{rrr|r} 7 & 7 & -7 & 0 \\ 0 & -7 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right],$$

add the second row to the first row,

$$\left[\begin{array}{rrr|r} 7 & 0 & -8 & 0 \\ 0 & -7 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right],$$

divide the first row by 7, and divide the second row by -7

$$\left[\begin{array}{rrr|r} 1 & 0 & -\frac{8}{7} & 0 \\ 0 & 1 & \frac{1}{7} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

The system of equations is now

$$\begin{cases} x - \frac{8}{7}z = 0, \\ y + \frac{1}{7}z = 0, \\ z = z. \end{cases}$$

Solving the system for x and y in terms of z gives us

$$\begin{cases} x = \frac{8}{7}z, \\ y = -\frac{1}{7}z, \\ z = z, \end{cases}$$

which can be written in matrix form as

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \frac{8}{7} \\ -\frac{1}{7} \\ 1 \end{bmatrix} z.$$

The specific solution in which $z = 0$ is the trivial solution,

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Another solution in which $z = 1$ is

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \frac{8}{7} \\ -\frac{1}{7} \\ 1 \end{bmatrix}.$$

5. Eigenvalues are the constants λ for a matrix A such that

$$Ax = \lambda x,$$

which means that

$$Ax = \lambda I x,$$
$$Ax - \lambda I x = 0,$$
$$(A - \lambda I)x = 0.$$

This equation implies a homogenous system of equations, which only has a non-trivial solution when the determinant is zero,

$$|A - \lambda I| = 0.$$

To find the eigenvalues, we will

- plug A into $(A - \lambda I)$ and write this matrix out,
- then write out the determinant,
- and solve for the values of λ that make the determinant zero.

A matrix is positive-definite if all of its eigenvalues are positive and negative-definite if all of its eigenvalues are negative. To find the unit eigenvectors associated with each matrix, we will

- plug each value of λ back in to $(A - \lambda I)x = 0$,
- write out the augmented matrix for this homogenous system,
- find a non-trivial solution,
- compute the magnitude of this solution vector,
- and divide the vector by its magnitude. This last step ensures that the eigenvector is a unit eigenvector.

(a) $\begin{bmatrix} -4 & 4 \\ 0 & -8 \end{bmatrix}$

$$A - \lambda I = \begin{bmatrix} -4 & 4 \\ 0 & -8 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -4-\lambda & 4 \\ 0 & -8-\lambda \end{bmatrix}$$

$$\begin{aligned} |A - \lambda I| &= (-4-\lambda)(-8-\lambda) - (4 \times 0) \\ &= (\lambda+8)(\lambda+4) = 0, \\ \lambda &= -8, -4. \end{aligned}$$

Since this matrix has only negative eigenvalues, it is negative-definite. The system of equations implied by $(A - \lambda I)x = 0$ is

$$\begin{cases} (-4-\lambda)x + 4y = 0, \\ -8 - \lambda y = 0 \end{cases}$$

with the augmented matrix

$$\left[\begin{array}{cc|c} -4-\lambda & 4 & 0 \\ 0 & -8-\lambda & 0 \end{array}\right].$$

Plugging in the first eigenvalue $\lambda = -8$ turns this augmented matrix into

$$\left[\begin{array}{cc|c} 4 & 4 & 0 \\ 0 & 0 & 0 \end{array}\right],$$

which we reduce with elementary row operations by dividing the first row by 4,

$$\left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right].$$

The system of equations is now

$$\begin{cases} x + y = 0, \\ y = y, \end{cases}$$

where y is a free variable. The system can be rewritten as

$$\begin{cases} x = -y, \\ y = y, \end{cases}$$

and in matrix form as

$$\begin{bmatrix} x \\ y \end{bmatrix} = y \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

One eigenvector that arises when $y = 1$ is $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$, which has magnitude

$$\left|\left| \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right|\right| = \sqrt{(-1)^2 + 1^2} = \sqrt{2}.$$

To find the unit eigenvector associated with $\lambda = -8$, we divide an eigenvector by its magnitude. In this case the unit eigenvector is

$$\begin{bmatrix} -\frac{1}{\sqrt{2}} \\ \\ \frac{1}{\sqrt{2}} \end{bmatrix}.$$

Plugging in the second eigenvalue $\lambda = -4$ turns the augmented matrix into

$$\left[\begin{array}{cc|c} 0 & 4 & 0 \\ 0 & -4 & 0 \end{array}\right],$$

which we reduce with elementary row operations by adding the second row to the first,

$$\left[\begin{array}{cc|c} 0 & 0 & 0 \\ 0 & -4 & 0 \end{array}\right],$$

then dividing the second row by -4,

$$\left[\begin{array}{cc|c} 0 & 0 & 0 \\ 0 & 1 & 0 \end{array}\right].$$

Since the first row corresponds to x, and it reduces to the obviously true statement 0=0, we think of x to be a free variable in which $x = x$. The system of equations is now

$$\begin{cases} x = x, \\ y = 0, \end{cases}$$

The system can be rewritten in matrix form as

$$\begin{bmatrix} x \\ y \end{bmatrix} = x \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

One eigenvector that arises when $x = 1$ is $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$, which has magnitude

$$\left|\left| \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right|\right| = \sqrt{1^2 + 0^2} = 1,$$

so it is the unit eigenvector associated with $\lambda = -4$.

(b) $\begin{bmatrix} -3 & 0 \\ 2 & 7 \end{bmatrix}$

$$A - \lambda I = \begin{bmatrix} -3 & 0 \\ 2 & 7 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -3-\lambda & 0 \\ 2 & 7-\lambda \end{bmatrix}$$

$$\begin{aligned} |A - \lambda I| &= (-3-\lambda)(7-\lambda) - (0 \times 2) \\ &= (\lambda+3)(\lambda-7) = 0, \\ \lambda &= -3,\, 7. \end{aligned}$$

Since this matrix has one positive and one negative eigenvalue, it is neither positive-definite nor negative-definite. The system of equations implied by $(A - \lambda I)x = 0$ is

$$\begin{cases} (-3-\lambda)x = 0, \\ 2x + (7-\lambda)y = 0 \end{cases}$$

with the augmented matrix

$$\left[\begin{array}{cc|c} -3-\lambda & 0 & 0 \\ 2 & 7-\lambda & 0 \end{array}\right].$$

Plugging in the first eigenvalue $\lambda = -3$ turns this augmented matrix into

$$\left[\begin{array}{cc|c} 0 & 0 & 0 \\ 2 & 10 & 0 \end{array}\right],$$

which we reduce with elementary row operations by interchanging the first and second rows,

$$\left[\begin{array}{cc|c} 2 & 10 & 0 \\ 0 & 0 & 0 \end{array}\right],$$

then dividing the first row by 2,

$$\left[\begin{array}{cc|c} 1 & 5 & 0 \\ 0 & 0 & 0 \end{array}\right].$$

The system of equations is now

$$\begin{cases} x + 5y = 0, \\ y = y, \end{cases}$$

where y is a free variable. The system can be rewritten as

$$\begin{cases} x = -5y, \\ y = y, \end{cases}$$

and in matrix form as

$$\begin{bmatrix} x \\ y \end{bmatrix} = y \begin{bmatrix} -5 \\ 1 \end{bmatrix}.$$

One eigenvector that arises when $y = 1$ is $\begin{bmatrix} -5 \\ 1 \end{bmatrix}$, which has magnitude

$$\left\| \begin{bmatrix} -5 \\ 1 \end{bmatrix} \right\| = \sqrt{(-5)^2 + 1^2} = \sqrt{26}.$$

To find the unit eigenvector associated with $\lambda = -3$, we divide an eigenvector by its magnitude. In this case the unit eigenvector is

$$\begin{bmatrix} -\frac{5}{\sqrt{26}} \\ \\ \frac{1}{\sqrt{26}} \end{bmatrix}.$$

Plugging in the second eigenvalue $\lambda = 7$ turns this augmented matrix into

$$\left[\begin{array}{cc|c} -10 & 0 & 0 \\ 2 & 0 & 0 \end{array}\right],$$

which we reduce with elementary row operations by dividing the first row by -10,

$$\left[\begin{array}{cc|c} 1 & 0 & 0 \\ 2 & 0 & 0 \end{array}\right],$$

then adding -2 times the first row to the second row,

$$\left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right].$$

The system of equations is now

$$\begin{cases} x = 0, \\ y = y, \end{cases}$$

where y is a free variable. The system can be rewritten in matrix form as

$$\begin{bmatrix} x \\ y \end{bmatrix} = y \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

One eigenvector that arises when $y = 1$ is $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$, which has magnitude

$$\left\| \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\| = \sqrt{0^2 + 1^2} = 1,$$

so this vector is the unit eigenvector associated with $\lambda = 7$.

(c) $\begin{bmatrix} 6 & 7 \\ -5 & -6 \end{bmatrix}$

$$A - \lambda I = \begin{bmatrix} 6 & 7 \\ -5 & -6 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 6-\lambda & 7 \\ -5 & -6-\lambda \end{bmatrix}$$

$$\begin{aligned} |A - \lambda I| &= (6-\lambda)(-6-\lambda) - (7 \times -5) \\ &= (\lambda - 6)(\lambda + 6) + 35 \\ &= \lambda^2 - 36 + 35 \\ &= \lambda^2 - 1 \\ &= (\lambda - 1)(\lambda + 1) = 0, \\ \lambda &= -1, 1. \end{aligned}$$

Since this matrix has one positive and one negative eigenvalue, it is neither positive-definite nor negative-definite. The system of equations implied by $(A - \lambda I)x = 0$ is

$$\begin{cases} (6-\lambda)x + 7y = 0, \\ -5x + (-6-\lambda)y = 0 \end{cases}$$

with the augmented matrix

$$\left[\begin{array}{cc|c} 6-\lambda & 7 & 0 \\ -5 & -6-\lambda & 0 \end{array}\right].$$

Plugging in the first eigenvalue $\lambda = -1$ turns this augmented matrix into

$$\left[\begin{array}{cc|c} 7 & 7 & 0 \\ -5 & -5 & 0 \end{array}\right],$$

which we reduce with elementary row operations by dividing the first row by 7,

$$\left[\begin{array}{rr|r} 1 & 1 & 0 \\ -5 & -5 & 0 \end{array}\right],$$

and adding 5 times the first row to the second,

$$\left[\begin{array}{rr|r} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right].$$

The system of equations is now

$$\begin{cases} x + y = 0, \\ y = y, \end{cases}$$

where y is a free variable. The system can be rewritten as

$$\begin{cases} x = -y, \\ y = y, \end{cases}$$

and in matrix form as

$$\begin{bmatrix} x \\ y \end{bmatrix} = y \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

One eigenvector that arises when $y = 1$ is $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$, which has magnitude

$$\left|\left| \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right|\right| = \sqrt{(-1)^2 + 1^2} = \sqrt{2}.$$

To find the unit eigenvector associated with $\lambda = -1$, we divide an eigenvector by its magnitude. In this case the unit eigenvector is

$$\begin{bmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}.$$

Plugging in the first eigenvalue $\lambda = 1$ turns this augmented matrix into

$$\left[\begin{array}{rr|r} 5 & 7 & 0 \\ -5 & -7 & 0 \end{array}\right],$$

which we reduce with elementary row operations by adding the first row to the second,

$$\left[\begin{array}{rr|r} 5 & 7 & 0 \\ 0 & 0 & 0 \end{array}\right],$$

and dividing the first row by 5,

$$\left[\begin{array}{rr|r} 1 & \frac{7}{5} & 0 \\ 0 & 0 & 0 \end{array}\right].$$

The system of equations is now

$$\begin{cases} x + \frac{7}{5}y = 0, \\ y = y, \end{cases}$$

where y is a free variable. The system can be rewritten as

$$\begin{cases} x = -\frac{7}{5}y, \\ y = y, \end{cases}$$

and in matrix form as

$$\begin{bmatrix} x \\ y \end{bmatrix} = y \begin{bmatrix} -\frac{7}{5} \\ \\ 1 \end{bmatrix}.$$

One eigenvector that arises when $y = 5$ is $\begin{bmatrix} -7 \\ 5 \end{bmatrix}$, which has magnitude

$$\left\| \begin{bmatrix} -7 \\ 5 \end{bmatrix} \right\| = \sqrt{(-7)^2 + 5^2} = \sqrt{74}.$$

To find the unit eigenvector associated with $\lambda = -1$, we divide an eigenvector by its magnitude. In this case the unit eigenvector is

$$\begin{bmatrix} -\frac{7}{\sqrt{74}} \\ \\ \frac{5}{\sqrt{74}} \end{bmatrix}.$$

(d) $\begin{bmatrix} 9 & 3 \\ 0 & -1 \end{bmatrix}$

$$A - \lambda I = \begin{bmatrix} 9 & 3 \\ 0 & -1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 9-\lambda & 3 \\ 0 & -1-\lambda \end{bmatrix}$$

$$\begin{aligned} |A - \lambda I| &= (9-\lambda)(-1-\lambda) - (3 \times 0) \\ &= (\lambda - 9)(\lambda + 1) = 0, \\ \lambda &= -1,\ 9. \end{aligned}$$

Since this matrix has one positive and one negative eigenvalue, it is neither positive-definite nor negative-definite. The system of equations implied by $(A - \lambda I)x = 0$ is

$$\begin{cases} (9-\lambda)x + 3y = 0, \\ (-1-\lambda)y = 0 \end{cases}$$

with the augmented matrix

$$\left[\begin{array}{cc|c} 9-\lambda & 3 & 0 \\ 0 & -1-\lambda & 0 \end{array}\right].$$

Plugging in the first eigenvalue $\lambda = -1$ turns this augmented matrix into

$$\left[\begin{array}{cc|c} 10 & 3 & 0 \\ 0 & 0 & 0 \end{array}\right],$$

which we reduce with elementary row operations by dividing the first row by 10,

$$\left[\begin{array}{cc|c} 1 & \frac{3}{10} & 0 \\ \\ 0 & 0 & 0 \end{array}\right].$$

The system of equations is now

$$\begin{cases} x + \frac{3}{10}y = 0, \\ y = y, \end{cases}$$

where y is a free variable. The system can be rewritten as

$$\begin{cases} x = -\frac{3}{10}y, \\ \\ y = y, \end{cases}$$

and in matrix form as

$$\begin{bmatrix} x \\ y \end{bmatrix} = y \begin{bmatrix} -\frac{3}{10} \\ 1 \end{bmatrix}.$$

One eigenvector that arises when $y = 10$ is $\begin{bmatrix} -3 \\ 10 \end{bmatrix}$, which has magnitude

$$\left|\left| \begin{bmatrix} -3 \\ 10 \end{bmatrix} \right|\right| = \sqrt{(-3)^2 + 10^2} = \sqrt{109}.$$

To find the unit eigenvector associated with $\lambda = -1$, we divide an eigenvector by its magnitude. In this case the unit eigenvector is

$$\begin{bmatrix} -\frac{3}{\sqrt{109}} \\ \\ \frac{10}{\sqrt{109}} \end{bmatrix}.$$

Plugging in the second eigenvalue $\lambda = 9$ turns this augmented matrix into

$$\left[\begin{array}{cc|c} 0 & 3 & 0 \\ 0 & -10 & 0 \end{array}\right],$$

which we reduce with elementary row operations by dividing the first row by 3 and the second row by -10,

$$\left[\begin{array}{cc|c} 0 & 1 & 0 \\ 0 & 1 & 0 \end{array}\right],$$

and adding -1 times the second row to the first,

$$\left[\begin{array}{cc|c} 0 & 0 & 0 \\ 0 & 1 & 0 \end{array}\right].$$

The system of equations is now

$$\begin{cases} x = x, \\ y = 0, \end{cases}$$

where x is the free variable. The system can be rewritten in matrix form as

$$\begin{bmatrix} x \\ y \end{bmatrix} = x \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

One eigenvector that arises when $x = 1$ is $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$, which has magnitude

$$\left|\left| \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right|\right| = \sqrt{1^2 + 0^2} = 1,$$

so this vector is the unit eigenvector associated with $\lambda = 9$.

(e) $\begin{bmatrix} -6 & -8 \\ -8 & -6 \end{bmatrix}$

$$A - \lambda I = \begin{bmatrix} -6 & -8 \\ -8 & -6 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -6-\lambda & -8 \\ -8 & -6-\lambda \end{bmatrix}$$

$$\begin{aligned} |A - \lambda I| &= (-6-\lambda)^2 - (-8)^2 \\ &= (\lambda + 6)^2 - 64 \\ &= (\lambda^2 + 12\lambda + 36) - 64 \\ &= \lambda^2 + 12\lambda - 28 \\ &= (\lambda + 14)(\lambda - 2) = 0, \\ \lambda &= -14,\ 2. \end{aligned}$$

Since this matrix has one positive and one negative eigenvalue, it is neither positive-definite nor negative-definite. The system of equations implied by $(A - \lambda I)x = 0$ is

$$\begin{cases} (-6-\lambda)x - 8y = 0, \\ -8x + (-6-\lambda)y = 0 \end{cases}$$

with the augmented matrix

$$\left[\begin{array}{rr|r} -6-\lambda & -8 & 0 \\ -8 & -6-\lambda & 0 \end{array}\right].$$

Plugging in the first eigenvalue $\lambda = -14$ turns this augmented matrix into

$$\left[\begin{array}{rr|r} 8 & -8 & 0 \\ -8 & 8 & 0 \end{array}\right],$$

which we reduce with elementary row operations by adding the first row to the second,

$$\left[\begin{array}{rr|r} 8 & -8 & 0 \\ 0 & 0 & 0 \end{array}\right],$$

and dividing the first row by 8,

$$\left[\begin{array}{rr|r} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right].$$

The system of equations is now

$$\begin{cases} x - y = 0, \\ y = y, \end{cases}$$

where y is a free variable. The system can be rewritten as

$$\begin{cases} x = y, \\ y = y, \end{cases}$$

and in matrix form as

$$\begin{bmatrix} x \\ y \end{bmatrix} = y \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

One eigenvector that arises when $y = 1$ is $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$, which has magnitude

$$\left\| \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\| = \sqrt{1^2 + 1^2} = \sqrt{2}.$$

To find the unit eigenvector associated with $\lambda = -14$, we divide an eigenvector by its magnitude. In this case the unit eigenvector is

$$\begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}.$$

Plugging in the second eigenvalue $\lambda = 2$ turns this augmented matrix into

$$\left[\begin{array}{rr|r} -8 & -8 & 0 \\ -8 & -8 & 0 \end{array}\right],$$

which we reduce with elementary row operations by adding the first row to the second,

$$\left[\begin{array}{rr|r} -8 & -8 & 0 \\ 0 & 0 & 0 \end{array}\right],$$

and dividing the first row by -8,

$$\left[\begin{array}{rr|r} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right].$$

The system of equations is now

$$\begin{cases} x + y = 0, \\ y = y, \end{cases}$$

where y is a free variable. The system can be rewritten as

$$\begin{cases} x = -y, \\ y = y, \end{cases}$$

and in matrix form as

$$\begin{bmatrix} x \\ y \end{bmatrix} = y \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

One eigenvector that arises when $y = 1$ is $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$, which has magnitude

$$\left\| \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\| = \sqrt{1^2 + 1^2} = \sqrt{2}.$$

To find the unit eigenvector associated with $\lambda = 2$, we divide an eigenvector by its magnitude. In this case the unit eigenvector is

$$\begin{bmatrix} -\frac{1}{\sqrt{2}} \\ \\ \frac{1}{\sqrt{2}} \end{bmatrix}.$$

(f) $\begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}$

$$A - \lambda I = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1-\lambda & 3 \\ 2 & 6-\lambda \end{bmatrix}$$

$$\begin{aligned} |A - \lambda I| &= (1-\lambda)(6-\lambda) - (3 \times 2) \\ &= (\lambda^2 - 7\lambda + 6) - 6 \\ &= \lambda^2 - 7\lambda \\ &= \lambda(\lambda - 7) = 0, \\ \lambda &= 0, 7. \end{aligned}$$

This matrix is not negative-definite because it has a negative eigenvalue. But it's also not positive-definite because 0 is not, strictly speaking, a positive number. We can say that this matrix is positive-*semi*definite. The system of equations implied by $(A - \lambda I)x = 0$ is

$$\begin{cases} (1-\lambda)x + 3y = 0, \\ 2x + (6-\lambda)y = 0 \end{cases}$$

with the augmented matrix

$$\left[\begin{array}{cc|c} 1-\lambda & 3 & 0 \\ 2 & 6-\lambda & 0 \end{array}\right].$$

Plugging in the first eigenvalue $\lambda = 0$ turns this augmented matrix into

$$\left[\begin{array}{cc|c} 1 & 3 & 0 \\ 2 & 6 & 0 \end{array}\right],$$

which we reduce with elementary row operations by adding -2 times the first row to the second,

$$\left[\begin{array}{cc|c} 1 & 3 & 0 \\ 0 & 0 & 0 \end{array}\right].$$

The system of equations is now

$$\begin{cases} x + 3y = 0, \\ y = y, \end{cases}$$

where y is a free variable. The system can be rewritten as

$$\begin{cases} x = -3y, \\ y = y, \end{cases}$$

and in matrix form as

$$\begin{bmatrix} x \\ y \end{bmatrix} = y \begin{bmatrix} -3 \\ 1 \end{bmatrix}.$$

One eigenvector that arises when $y = 1$ is $\begin{bmatrix} -3 \\ 1 \end{bmatrix}$, which has magnitude

$$\left\| \begin{bmatrix} -3 \\ 1 \end{bmatrix} \right\| = \sqrt{(-3)^2 + 1^2} = \sqrt{10}.$$

To find the unit eigenvector associated with $\lambda = 0$, we divide an eigenvector by its magnitude. In this case the unit eigenvector is

$$\begin{bmatrix} -\frac{3}{\sqrt{10}} \\ \\ \frac{1}{\sqrt{10}} \end{bmatrix}.$$

In other words, the matrix $\begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}$, when left-multiplied by any vector that is a multiple of $\begin{bmatrix} -\frac{3}{\sqrt{10}} \\ \\ \frac{1}{\sqrt{10}} \end{bmatrix}$, always outputs a vector of all zeroes.

Plugging in the second eigenvalue $\lambda = 7$ turns this augmented matrix into

$$\left[\begin{array}{rr|r} -6 & 3 & 0 \\ 2 & -1 & 0 \end{array}\right],$$

which we reduce with elementary row operations by dividing the first row by -3,

$$\left[\begin{array}{rr|r} 2 & -1 & 0 \\ 2 & -1 & 0 \end{array}\right],$$

then multiplying the first row by -1 and adding it to the second row,

$$\left[\begin{array}{rr|r} 2 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right],$$

and finally dividing the first row by 2,

$$\left[\begin{array}{rr|r} 1 & -\frac{1}{2} & 0 \\ \\ 0 & 0 & 0 \end{array}\right].$$

The system of equations is now

$$\begin{cases} x - \frac{1}{2}y = 0, \\ \\ y = y, \end{cases}$$

where y is a free variable. The system can be rewritten as

$$\begin{cases} x = \frac{1}{2}y, \\ \\ y = y, \end{cases}$$

and in matrix form as

$$\begin{bmatrix} x \\ y \end{bmatrix} = y \begin{bmatrix} \frac{1}{2} \\ \\ 1 \end{bmatrix}.$$

One eigenvector that arises when $y = 2$ is $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$, which has magnitude

$$\left\| \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\| = \sqrt{1^2 + 2^2} = \sqrt{5}.$$

To find the unit eigenvector associated with $\lambda = 7$, we divide an eigenvector by its magnitude. In this case the unit eigenvector is

$$\begin{bmatrix} \frac{1}{\sqrt{5}} \\ \\ \frac{2}{\sqrt{5}} \end{bmatrix}.$$

(g) $\begin{bmatrix} 8 & -10 \\ 0 & -9 \end{bmatrix}$

$$A - \lambda I = \begin{bmatrix} 8 & -10 \\ 0 & -9 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 8 - \lambda & -10 \\ 0 & -9 - \lambda \end{bmatrix}$$

$$\begin{aligned} |A - \lambda I| &= (8 - \lambda)(-9 - \lambda) - (-10 \times 0) \\ &= (\lambda - 8)(\lambda + 9) = 0, \\ \lambda &= -9, 8. \end{aligned}$$

Since this matrix has one positive and one negative eigenvalue, it is neither positive-definite nor negative-definite. The system of equations implied by $(A - \lambda I)x = 0$ is

$$\begin{cases} (8 - \lambda)x - 10y = 0, \\ (-9 - \lambda)y = 0 \end{cases}$$

with the augmented matrix

$$\left[\begin{array}{cc|c} 8 - \lambda & -10 & 0 \\ 0 & -9 - \lambda & 0 \end{array}\right].$$

Plugging in the first eigenvalue $\lambda = -9$ turns this augmented matrix into

$$\left[\begin{array}{cc|c} 17 & -10 & 0 \\ 0 & 0 & 0 \end{array}\right],$$

which we reduce with elementary row operations by dividing the first row by 17,

$$\left[\begin{array}{cc|c} 1 & -\frac{10}{17} & 0 \\ \\ 0 & 0 & 0 \end{array}\right].$$

The system of equations is now

$$\begin{cases} x - \frac{10}{17}y = 0, \\ \\ y = y, \end{cases}$$

where y is a free variable. The system can be rewritten as

$$\begin{cases} x = \frac{10}{17}y, \\ \\ y = y, \end{cases}$$

and in matrix form as

$$\begin{bmatrix} x \\ y \end{bmatrix} = y \begin{bmatrix} \frac{10}{17} \\ \\ 1 \end{bmatrix}.$$

One eigenvector that arises when $y = 17$ is $\begin{bmatrix} 10 \\ 17 \end{bmatrix}$, which has magnitude

$$\left| \begin{bmatrix} 10 \\ 17 \end{bmatrix} \right| = \sqrt{10^2 + 17^2} = \sqrt{389}.$$

To find the unit eigenvector associated with $\lambda = -9$, we divide an eigenvector by its magnitude. In this case the unit eigenvector is

$$\begin{bmatrix} \frac{10}{\sqrt{389}} \\ \\ \frac{17}{\sqrt{389}} \end{bmatrix}.$$

Plugging in the second eigenvalue $\lambda = 8$ turns this augmented matrix into

$$\left[\begin{array}{cc|c} 0 & -10 & 0 \\ 0 & -17 & 0 \end{array}\right],$$

which we reduce with elementary row operations by dividing the second row by -17,

$$\left[\begin{array}{cr|c} 0 & -10 & 0 \\ 0 & 1 & 0 \end{array}\right],$$

and adding 10 times the second row to the first row,

$$\left[\begin{array}{cc|c} 0 & 0 & 0 \\ 0 & 1 & 0 \end{array}\right].$$

The system of equations is now

$$\begin{cases} x = x, \\ y = 0, \end{cases}$$

where x is the free variable. The system can be rewritten in matrix form as

$$\begin{bmatrix} x \\ y \end{bmatrix} = x \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

One eigenvector that arises when $x = 1$ is $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$, which has magnitude

$$\left| \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right| = \sqrt{1^2 + 0^2} = 1,$$

so this vector is the unit eigenvector associated with $\lambda = 8$.

(h) $\begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix}$

$$A - \lambda I = \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 7-\lambda & 0 \\ 0 & 7-\lambda \end{bmatrix}$$

$$\begin{aligned} |A - \lambda I| &= (7-\lambda)^2 - (0 \times 0) \\ &= (7-\lambda)^2 = 0, \\ \lambda &= 7. \end{aligned}$$

Since this matrix has only positive eigenvalues, it is positive-definite. The system of equations implied by $(A - \lambda I)x = 0$ is

$$\begin{cases} (7-\lambda)x = 0, \\ (7-\lambda)y = 0 \end{cases}$$

with the augmented matrix

$$\left[\begin{array}{rr|r} 7-\lambda & 0 & 0 \\ 0 & 7-\lambda & 0 \end{array}\right].$$

Plugging in the only eigenvalue $\lambda = 7$ turns this augmented matrix into

$$\left[\begin{array}{rr|r} 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right],$$

which cannot be reduced. The system of equations is now

$$\begin{cases} x = x, \\ y = y, \end{cases}$$

where x and y are both free variables. The system can be rewritten in matrix form as

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix},$$

which means that **any** two-dimensional, real-numbered vector is an eigenvector of this matrix associated with the eigenvalue $\lambda = 7$. To find the unit eigenvectors, we write out the magnitude:

$$\left\| \begin{bmatrix} x \\ y \end{bmatrix} \right\| = \sqrt{x^2 + y^2}.$$

The unit eigenvalues are all of the form

$$\begin{bmatrix} \frac{x}{\sqrt{x^2+y^2}} \\ \\ \frac{y}{\sqrt{x^2+y^2}} \end{bmatrix},$$

which is defined for any values of x and y other than $(0, 0)$ since that would divide each element by 0. For example, the unit eigenvector we derive when $x = 2$ and $y = 5$ is

$$\begin{bmatrix} \frac{2}{\sqrt{2^2+5^2}} \\ \\ \frac{5}{\sqrt{2^2+5^2}} \end{bmatrix} = \begin{bmatrix} \frac{2}{\sqrt{29}} \\ \\ \frac{5}{\sqrt{29}} \end{bmatrix}.$$

(i) $\begin{bmatrix} 0 & -3 \\ -6 & -4 \end{bmatrix}$

$$A - \lambda I = \begin{bmatrix} 0 & -3 \\ -6 & -4 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0-\lambda & -3 \\ -6 & -4-\lambda \end{bmatrix}$$

$$\begin{aligned} |A - \lambda I| &= (-\lambda)(-4-\lambda) - (-3 \times -6) \\ &= 4\lambda + \lambda^2 - 18 \\ &= \lambda^2 + 4\lambda - 18 = 0. \end{aligned}$$

$$\lambda = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-4 \pm \sqrt{16 - 4(1)(-18)}}{2}$$

$$= \frac{-4 \pm \sqrt{88}}{2} = \frac{-4 \pm 2\sqrt{22}}{2} = -2 \pm \sqrt{22},$$

$$\lambda = -2 - \sqrt{22} = -6.69, \qquad \lambda = -2 + \sqrt{22} = 2.69.$$

Since this matrix has one positive and one negative eigenvalue, it is neither positive-definite nor negative-definite. The system of equations implied by $(A - \lambda I)x = 0$ is

$$\begin{cases} -\lambda x - 3y = 0, \\ -6x + (-4 - \lambda)y = 0 \end{cases}$$

with the augmented matrix

$$\left[\begin{array}{rr|r} -\lambda & -3 & 0 \\ -6 & -4-\lambda & 0 \end{array}\right].$$

Plugging in the first eigenvalue $\lambda = -6.69$ turns this augmented matrix into

$$\left[\begin{array}{rr|r} 6.69 & -3 & 0 \\ -6 & 2.69 & 0 \end{array}\right],$$

which we reduce with elementary row operations by diving the first row by 6.69,

$$\left[\begin{array}{rr|r} 1 & -.45 & 0 \\ -6 & 2.69 & 0 \end{array}\right],$$

dividing the second row by -6,

$$\left[\begin{array}{rr|r} 1 & -.45 & 0 \\ 1 & -.45 & 0 \end{array}\right],$$

and adding -1 times the first row to the second row,

$$\left[\begin{array}{rr|r} 1 & -.45 & 0 \\ 0 & 0 & 0 \end{array}\right].$$

The system of equations is now

$$\begin{cases} x - .45y = 0, \\ y = y, \end{cases}$$

where y is a free variable. The system can be rewritten as

$$\begin{cases} x = .45y, \\ y = y, \end{cases}$$

and in matrix form as

$$\left[\begin{array}{c} x \\ y \end{array}\right] = y \left[\begin{array}{r} .45 \\ 1 \end{array}\right].$$

One eigenvector that arises when $y = 1$ is $\left[\begin{array}{r} .45 \\ 1 \end{array}\right]$, which has magnitude

$$\left\| \left[\begin{array}{r} .45 \\ 1 \end{array}\right] \right\| = \sqrt{.45^2 + 1^2} = \sqrt{1.2} = 1.095.$$

To find the unit eigenvector associated with $\lambda = -6.69$, we divide an eigenvector by its magnitude. In this case the unit eigenvector is

$$\left[\begin{array}{c} \frac{.45}{1.095} \\ \\ \frac{1}{1.095} \end{array}\right] = \left[\begin{array}{c} .41 \\ .91 \end{array}\right].$$

Plugging in the second eigenvalue $\lambda = 2.69$ turns this augmented matrix into

$$\left[\begin{array}{rr|r} -2.69 & -3 & 0 \\ -6 & -6.69 & 0 \end{array}\right],$$

which we reduce with elementary row operations by diving the first row by -2.69,

$$\left[\begin{array}{cc|c} 1 & 1.12 & 0 \\ -6 & -5.69 & 0 \end{array}\right],$$

dividing the second row by -6,

$$\left[\begin{array}{cc|c} 1 & 1.12 & 0 \\ 1 & 1.12 & 0 \end{array}\right],$$

and adding -1 times the first row to the second row,

$$\left[\begin{array}{cc|c} 1 & 1.12 & 0 \\ 0 & 0 & 0 \end{array}\right].$$

The system of equations is now

$$\begin{cases} x + 1.12y = 0, \\ y = y, \end{cases}$$

where y is a free variable. The system can be rewritten as

$$\begin{cases} x = -1.12y, \\ y = y, \end{cases}$$

and in matrix form as

$$\begin{bmatrix} x \\ y \end{bmatrix} = y \begin{bmatrix} -1.12 \\ 1 \end{bmatrix}.$$

One eigenvector that arises when $y = 1$ is $\begin{bmatrix} -1.12 \\ 1 \end{bmatrix}$, which has magnitude

$$\left|\left| \begin{bmatrix} -1.12 \\ 1 \end{bmatrix} \right|\right| = \sqrt{(-1.12)^2 + 1^2} = \sqrt{2.25} = 1.5.$$

To find the unit eigenvector associated with $\lambda = 2.69$, we divide an eigenvector by its magnitude. In this case the unit eigenvector is

$$\begin{bmatrix} -\frac{1.12}{1.5} \\ \\ \frac{1}{1.5} \end{bmatrix} = \begin{bmatrix} -.75 \\ .67 \end{bmatrix}.$$

(j) $\begin{bmatrix} -5 & 2 \\ 2 & 6 \end{bmatrix}$

$$A - \lambda I = \begin{bmatrix} -5 & 2 \\ 2 & 6 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -5-\lambda & 2 \\ 2 & 6-\lambda \end{bmatrix}$$

$$\begin{aligned} |A - \lambda I| &= (-5-\lambda)(6-\lambda) - (2 \times 2) \\ &= (\lambda + 5)(\lambda - 6) - 4 \\ &= (\lambda^2 - \lambda - 30) + 4 \\ &= \lambda^2 - \lambda - 26 = 0. \end{aligned}$$

$$\lambda = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{1 \pm \sqrt{1 - 4(1)(-26)}}{2} = \frac{1 \pm \sqrt{105}}{2}.$$

$$\lambda = \frac{1 - \sqrt{105}}{2} = -5.35, \qquad \lambda = \frac{1 + \sqrt{105}}{2} = 6.35.$$

Since this matrix has one positive and one negative eigenvalue, it is neither positive-definite nor negative-definite. The system of equations implied by $(A - \lambda I)x = 0$ is

$$\begin{cases} (-5-\lambda)x + 2y = 0, \\ 2x + (6-\lambda)y = 0 \end{cases}$$

with the augmented matrix

$$\left[\begin{array}{rr|r} -5-\lambda & 2 & 0 \\ 2 & 6-\lambda & 0 \end{array}\right].$$

Plugging in the first eigenvalue $\lambda = -5.35$ turns this augmented matrix into

$$\left[\begin{array}{rr|r} .35 & 2 & 0 \\ 2 & 11.35 & 0 \end{array}\right],$$

which we reduce with elementary row operations by diving the first row by .35,

$$\left[\begin{array}{rr|r} 1 & 5.7 & 0 \\ 2 & 11.35 & 0 \end{array}\right],$$

dividing the second row by 2,

$$\left[\begin{array}{rr|r} 1 & 5.7 & 0 \\ 1 & 5.7 & 0 \end{array}\right],$$

and adding -1 times the first row to the second row,

$$\left[\begin{array}{rr|r} 1 & 5.7 & 0 \\ 0 & 0 & 0 \end{array}\right].$$

The system of equations is now

$$\begin{cases} x + 5.7y = 0, \\ y = y, \end{cases}$$

where y is a free variable. The system can be rewritten as

$$\begin{cases} x = -5.7y, \\ y = y, \end{cases}$$

and in matrix form as

$$\begin{bmatrix} x \\ y \end{bmatrix} = y \begin{bmatrix} -5.7 \\ 1 \end{bmatrix}.$$

One eigenvector that arises when $y = 1$ is $\begin{bmatrix} -5.7 \\ 1 \end{bmatrix}$, which has magnitude

$$\left|\left| \begin{bmatrix} -5.7 \\ 1 \end{bmatrix} \right|\right| = \sqrt{(-5.7)^2 + 1^2} = \sqrt{33.49} = 5.79.$$

To find the unit eigenvector associated with $\lambda = -5.35$, we divide an eigenvector by its magnitude. In this case the unit eigenvector is

$$\begin{bmatrix} -\frac{5.7}{5.79} \\ \\ \frac{1}{5.79} \end{bmatrix} = \begin{bmatrix} -.98 \\ .17 \end{bmatrix}.$$

Plugging in the second eigenvalue $\lambda = 6.35$ turns this augmented matrix into

$$\left[\begin{array}{rr|r} -11.35 & 2 & 0 \\ 2 & -.35 & 0 \end{array}\right],$$

which we reduce with elementary row operations by diving the first row by -11.35,

$$\left[\begin{array}{rr|r} 1 & -.18 & 0 \\ 2 & -.35 & 0 \end{array}\right],$$

dividing the second row by 2,

$$\left[\begin{array}{cc|c} 1 & -.18 & 0 \\ 1 & -.18 & 0 \end{array}\right],$$

and adding -1 times the first row to the second row,

$$\left[\begin{array}{cc|c} 1 & -.18 & 0 \\ 0 & 0 & 0 \end{array}\right].$$

The system of equations is now

$$\begin{cases} x - .18y = 0, \\ y = y, \end{cases}$$

where y is a free variable. The system can be rewritten as

$$\begin{cases} x = .18y, \\ y = y, \end{cases}$$

and in matrix form as

$$\begin{bmatrix} x \\ y \end{bmatrix} = y \begin{bmatrix} .18 \\ 1 \end{bmatrix}.$$

One eigenvector that arises when $y = 1$ is $\begin{bmatrix} .18 \\ 1 \end{bmatrix}$, which has magnitude

$$\left|\left| \begin{bmatrix} .18 \\ 1 \end{bmatrix} \right|\right| = \sqrt{(.18)^2 + 1^2} = \sqrt{1.03} = 1.02.$$

To find the unit eigenvector associated with $\lambda = 6.35$, we divide an eigenvector by its magnitude. In this case the unit eigenvector is

$$\begin{bmatrix} \frac{.18}{1.02} \\ \\ \frac{1}{1.02} \end{bmatrix} = \begin{bmatrix} .17 \\ .98 \end{bmatrix}.$$

6. (a) To find the gradient, we first take the partial derivative with respect to x,

$$\begin{aligned} \frac{\partial f}{\partial x} &= \frac{\partial}{\partial x}\left(xy + 2x + y + x^2 + 2y^3\right) \\ &= y + 2 + 2x, \end{aligned}$$

and the partial derivative with respect to y,

$$\begin{aligned} \frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\left(xy + 2x + y + x^2 + 2y^3\right) \\ &= x + 1 + 6y^2. \end{aligned}$$

The gradient is therefore

$$\nabla f(x, y) = \begin{bmatrix} y + 2 + 2x \\ x + 1 + 6y^2 \end{bmatrix}.$$

(b) The Hessian is given by the (2×2) matrix

$$H\Big(f(x, y)\Big) = \begin{bmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \\ \frac{\partial^2 f}{\partial x \partial y} & \frac{\partial^2 f}{\partial y^2} \end{bmatrix}.$$

To find the (1,1) element, the second partial derivative with respect to x and then x again, we take the partial derivative of $\frac{\partial f}{\partial x}$ with respect to x:

$$\frac{\partial f}{\partial x}\left(y + 2 + 2x\right) = 2.$$

To find the (1,2) and (2,1) elements, the second partial derivative with respect to x and then y, we either take the partial derivative of $\frac{\partial f}{\partial x}$ with respect to y, or the partial derivative of $\frac{\partial f}{\partial y}$ with respect to x. These two derivatives are both equal to:

$$\frac{\partial f}{\partial y}\left(y + 2 + 2x\right) = 1.$$

Finally, to find the (2,2) element, the second partial derivative with respect to y and then y again, we take the partial derivative of $\frac{\partial f}{\partial y}$ with respect to y:

$$\frac{\partial f}{\partial y}\left(x + 1 + 6y^2\right) = 12y.$$

So the Hessian is

$$H\left(f(x,y)\right) = \begin{bmatrix} 2 & 1 \\ 1 & 12y \end{bmatrix}.$$

(c) To find the critical points, we set each element of the gradient simultaneously equal to 0, and solve for the values of x and y that make this true. The system of equations is

$$\begin{cases} y + 2 + 2x = 0, \\ x + 1 + 6y^2 = 0. \end{cases}$$

There are many ways to solve this system, but here's one approach. We start by solving the first equation for y,

$$y = -2x - 2,$$

and substituting for y in the second equation,

$$x + 1 + 6(-2x - 2)^2 = 0,$$

$$\begin{aligned} x + 1 + 6(4x^2 + 8x + 4) &= 0, \\ x + 1 + 24x^2 + 48x + 24 &= 0, \\ 24x^2 + 49x + 25 &= 0, \end{aligned}$$

We can factor this quadratic expression according to the steps outlined in section 1.7.2. This quadratic expression is $ax^2 + bx + c$ where $a = 24$, $b = 49$, and $c = 25$. First we multiply a and c together

$$a \times c = 600,$$

find all pairs of integer factors that also multiply to this product:

$$\begin{array}{cccccc} 1 \times 600, & 2 \times 300, & 3 \times 200, & 4 \times 150, & 5 \times 120, & 6 \times 100, \\ 8 \times 75, & 10 \times 60, & 12 \times 50, & 15 \times 40, & 20 \times 30, & 24 \times 25, \end{array}$$

and look for a pair that adds to $b = 49$. In this case such a pair is 24 and 25. Then we break the middle term of the quadratic expression into two addends equal to these two factors,

$$24x^2 + (24x + 25x) + 25 = 0,$$

place parentheses around the first two and last two terms,

$$(24x^2 + 24x) + (25x + 25) = 0,$$

and pull all common factors outside each set of parentheses,

$$24x(x + 1) + 25(x + 1) = 0.$$

Finally we pull the common parenthetical factor out of both terms,

$$(x + 1)(24x + 25) = 0.$$

The values of x that solve this equation are -1 and $-\frac{25}{24}$. When $x = -1$, then y equals

$$y = -2(-1) - 2 = 0,$$

and when $x = -\frac{25}{24}$, y equals

$$y = -2\Big(-\frac{25}{24}\Big) - 2 = \frac{25}{12} - \frac{24}{12} = \frac{1}{12}.$$

Therefore the points $(x, y) = (-1, 0)$ and $(x, y) = \Big(\frac{25}{12}, \frac{1}{12}\Big)$ are critical points for the function.

(d) A critical point represents a local maximum if the Hessian is negative-definite after plugging in the critical point, and the critical point represents a local minimum if the Hessian is positive-definite after plugging in the critical point. If the Hessian is neither negative-definite nor positive-definite, then the critical point represents a saddle point. A matrix is negative-definite when all of its eigenvalues are negative, and positive-definite when all of its eigenvalues are positive. So here we have to first plug each critical point we found in part (c) into Hessian we found in part (b), then find the eigenvalues of the resulting matrix, and check whether the eigenvalues are all negative or all positive.

Plugging the critical point (-1,0) into the Hessian gives us

$$H\Big(f(-1,0)\Big) = \begin{bmatrix} 2 & 1 \\ 1 & 12(0) \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}.$$

To find the eigenvalues, we write

$$A - \lambda I = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2-\lambda & 1 \\ 1 & -\lambda \end{bmatrix},$$

and find the values of λ that make the determinant of this matrix equal to 0,

$$\begin{aligned} |A - \lambda I| &= (2-\lambda)(-\lambda) - (1 \times 1) \\ &= \lambda^2 - 2\lambda - 1 = 0. \end{aligned}$$

In this case we have to use the quadratic formula to solve for λ,

$$\begin{aligned} \lambda = &\frac{2 \pm \sqrt{4 - 4(1)(-1)}}{2}, \\ &= \frac{2 \pm \sqrt{8}}{2}, \\ &= \frac{2 \pm 2\sqrt{2}}{2}, \end{aligned}$$

$$= 1 \pm \sqrt{2}.$$

$$\lambda = 1 - \sqrt{2} = -.42, \qquad \lambda = 1 + \sqrt{2} = 2.42.$$

Since this matrix has one positive and one negative eigenvalue, it is neither negative-definite nor positive-definite and the critical point (-1,0) represents a saddle point.

Plugging the critical point $\left(\frac{25}{12}, \frac{1}{12}\right)$ into the Hessian gives us

$$H\Big(f(-1,0)\Big) = \begin{bmatrix} 2 & 1 \\ 1 & 12\left(\frac{1}{12}\right) \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}.$$

To find the eigenvalues, we write

$$A - \lambda I = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2-\lambda & 1 \\ 1 & 1-\lambda \end{bmatrix},$$

and find the values of λ that make the determinant of this matrix equal to 0,

$$\begin{aligned} |A - \lambda I| &= (2-\lambda)(1-\lambda) - (1 \times 1) \\ &= (\lambda^2 - 3\lambda + 2) - 1 \\ &= \lambda^2 - 3\lambda + 1 = 0. \end{aligned}$$

In this case we again have to use the quadratic formula to solve for λ,

$$\lambda = \frac{3 \pm \sqrt{9 - 4(1)(1)}}{2},$$

$$= \frac{3 \pm \sqrt{5}}{2}.$$

$$\lambda = \frac{3 - \sqrt{5}}{2} = .38, \qquad \lambda = \frac{3 + \sqrt{5}}{2} = 2.62.$$

Since all of the eigenvalues of this matrix are positive, it is negative-definite and the critical point $\left(\frac{25}{12}, \frac{1}{12}\right)$ represents a local minimum.

7. (a) First, we find the trace, determinant, and eigenvalues of each of the 6 matrices:

- $\begin{bmatrix} 3 & 0 \\ 6 & 4 \end{bmatrix}$

  The trace is 3+4=7.
  The determinant is $(3 \times 4) - (0 \times 6)$=12.
  The eigenvalues are the values λ that make the determinant of $\begin{bmatrix} 3-\lambda & 0 \\ 6 & 4-\lambda \end{bmatrix}$ equal 0. This characteristic equation is:

$$(3-\lambda)(4-\lambda) - (0 \times 6) = 0,$$

$$(\lambda - 3)(\lambda - 4) = 0,$$

$$\lambda = 3,\ 4.$$

- $\begin{bmatrix} 5 & -4 \\ -7 & 2 \end{bmatrix}$

  The trace is 5+2=7.
  The determinant is $(5 \times 2) - (-4 \times -7)$=-18.
  The eigenvalues are the values λ that make the determinant of $\begin{bmatrix} 5-\lambda & -4 \\ -7 & 2-\lambda \end{bmatrix}$ equal 0. This characteristic equation is:

$$(5-\lambda)(2-\lambda) - (-4 \times -7) = 0,$$
$$(\lambda-5)(\lambda-2) - 28 = 0,$$
$$(\lambda^2 - 7\lambda + 10) - 28 = 0,$$
$$\lambda^2 - 7\lambda - 18 = 0,$$
$$(\lambda-9)(\lambda+2) = 0,$$
$$\lambda = 9, -2.$$

- $\begin{bmatrix} -2 & 2 \\ 9 & 5 \end{bmatrix}$

  The trace is -2+5=3.
  The determinant is $(-2 \times 5) - (2 \times 9)$=-28.
  The eigenvalues are the values λ that make the determinant of $\begin{bmatrix} -2-\lambda & 2 \\ 9 & 5-\lambda \end{bmatrix}$ equal 0. This characteristic equation is:

$$(-2-\lambda)(5-\lambda) - (2 \times 9) = 0,$$
$$(\lambda+2)(\lambda-5) - 18 = 0,$$
$$(\lambda^2 - 3\lambda - 10) - 18 = 0,$$
$$\lambda^2 - 3\lambda - 28 = 0,$$
$$(\lambda-7)(\lambda+4) = 0,$$
$$\lambda = -4, 7.$$

- $\begin{bmatrix} 2 & -5 \\ -4 & 2 \end{bmatrix}$

  The trace is 2+2=4.
  The determinant is $(2 \times 2) - (-4 \times -5)$=-16.
  The eigenvalues are the values λ that make the determinant of $\begin{bmatrix} 2-\lambda & -5 \\ -4 & 2-\lambda \end{bmatrix}$ equal 0. This characteristic equation is:

$$(2-\lambda)(2-\lambda) - (-4 \times -5) = 0,$$
$$(\lambda-2)^2 - 20 = 0,$$
$$(\lambda^2 - 4\lambda + 4) - 20 = 0,$$
$$\lambda^2 - 4\lambda - 16 = 0,$$
$$\lambda = \frac{4 \pm \sqrt{16 - 4(1)(-16)}}{2},$$

$$
\begin{aligned}
&= \frac{4 \pm \sqrt{16+64}}{2}, \\
&= \frac{4 \pm \sqrt{80}}{2}, \\
&= \frac{4 \pm 4\sqrt{5}}{2}, \\
\lambda &= (2-2\sqrt{5}),\ (2+2\sqrt{5}).
\end{aligned}
$$

- $\begin{bmatrix} -10 & 9 \\ 6 & 2 \end{bmatrix}$

  The trace is -10+2=-8.
  The determinant is $(-10 \times 2) - (6 \times 9)$=-74.
  The eigenvalues are the values λ that make the determinant of $\begin{bmatrix} -10-\lambda & 9 \\ 6 & 2-\lambda \end{bmatrix}$ equal 0. This characteristic equation is:

$$
\begin{aligned}
(-10-\lambda)(2-\lambda) - (6 \times 9) &= 0, \\
(\lambda+10)(\lambda-2) - 54 &= 0, \\
(\lambda^2 + 8\lambda - 20) - 54 &= 0, \\
\lambda^2 + 8\lambda - 74 &= 0, \\
\lambda &= \frac{-8 \pm \sqrt{64 - 4(1)(-74)}}{2}, \\
&= \frac{-8 \pm \sqrt{64+296}}{2}, \\
&= \frac{-8 \pm 6\sqrt{10}}{2}, \\
\lambda &= (-4-3\sqrt{10}),\ (-4+3\sqrt{10}).
\end{aligned}
$$

- $\begin{bmatrix} 5 & 4 \\ -2 & -6 \end{bmatrix}$

  The trace is 5-6=-1.
  The determinant is $(5 \times -6) - (-2 \times 4)$=-22.
  The eigenvalues are the values λ that make the determinant of $\begin{bmatrix} 5-\lambda & 4 \\ -2 & -6-\lambda \end{bmatrix}$ equal 0. This characteristic equation is:

$$
\begin{aligned}
(5-\lambda)(-6-\lambda) - (-2 \times 4) &= 0, \\
(\lambda-5)(\lambda+6) + 8 &= 0, \\
(\lambda^2 + \lambda - 30) + 8 &= 0, \\
\lambda^2 + \lambda - 22 &= 0, \\
\lambda &= \frac{-1 \pm \sqrt{1 - 4(1)(-22)}}{2},
\end{aligned}
$$

$$= \frac{-1 \pm \sqrt{1+88}}{2},$$
$$= \frac{-1 \pm \sqrt{89}}{2},$$
$$\lambda = \left(-\frac{1}{2} - \frac{\sqrt{89}}{2}\right), \left(-\frac{1}{2} + \frac{\sqrt{89}}{2}\right).$$

(b) The first three matrices show a clear pattern: the trace is the **sum of the eigenvalues** and the determinant is the **product of the eigenvalues**. With a little work we can show that these properties hold for the last three matrices as well, when the eigenvalues are not integers. Consider the fourth matrix. The sum of the eigenvalues is

$$(2 - 2\sqrt{5}) + (2 + 2\sqrt{5}) = 4,$$

which is the trace, and the product of the eigenvalues is[1]

$$(2 - 2\sqrt{5})(2 + 2\sqrt{5}) = 4 - (2\sqrt{5})^2 = 4 - 4(5) = -16,$$

which is the determinant. Next consider the fifth matrix. The sum of the eigenvalues is

$$(-4 - 3\sqrt{10}) + (-4 + 3\sqrt{10}) = -8,$$

which is the trace, and the product of the eigenvalues is

$$(-4 - 3\sqrt{10})(-4 + 3\sqrt{10}) = 16 - 9(10) = -74,$$

which is the determinant. Finally, consider the last matrix. The sum of the eigenvalues is

$$\left(-\frac{1}{2} - \frac{\sqrt{89}}{2}\right) + \left(-\frac{1}{2} + \frac{\sqrt{89}}{2}\right) = -1,$$

which is the trace, and the product of the eigenvalues is

$$\left(-\frac{1}{2} - \frac{\sqrt{89}}{2}\right)\left(-\frac{1}{2} + \frac{\sqrt{89}}{2}\right) = \frac{1}{4} - \frac{89}{4} = -\frac{88}{4} = -22,$$

which is again the determinant.

8. This problem asks us to demonstrate that the product QBQ^{-1}, where

$$Q = \begin{bmatrix} 0.91 & 0.35 & -0.38 & 0.26 \\ 0.28 & -0.44 & 0.00 & 0.64 \\ -0.27 & 0.83 & -0.44 & 0.03 \\ 0.11 & 0.03 & 0.81 & -0.72 \end{bmatrix}, \qquad B = \begin{bmatrix} 10.47 & 0 & 0 & 0 \\ 0 & -9.21 & 0 & 0 \\ 0 & 0 & -7.65 & 0 \\ 0 & 0 & 0 & -3.60 \end{bmatrix},$$

and

$$Q^{-1} = \begin{bmatrix} 0.97 & -0.18 & -0.52 & 0.18 \\ 0.09 & 0.92 & 1.62 & 0.94 \\ -0.45 & 2.00 & 1.19 & 1.68 \\ -0.35 & 2.27 & 1.33 & 0.57 \end{bmatrix},$$

[1] It's easiest to use the difference of squares formula to evaluate this product: $(a+b)(a-b) = a^2 - b^2$.

is equal to

$$A = \begin{bmatrix} 8 & -1 & -8 & 3 \\ 4 & -2 & 2 & 3 \\ -5 & 0 & -7 & -2 \\ 3 & -7 & -5 & -9 \end{bmatrix}.$$

All we have to do is compute the matrix product QBQ^{-1}. First, consider the two left factors QB:

$$QB = \begin{bmatrix} 0.91 & 0.35 & -0.38 & 0.26 \\ 0.28 & -0.44 & 0.00 & 0.64 \\ -0.27 & 0.83 & -0.44 & 0.03 \\ 0.11 & 0.03 & 0.81 & -0.72 \end{bmatrix} \begin{bmatrix} 10.47 & 0 & 0 & 0 \\ 0 & -9.21 & 0 & 0 \\ 0 & 0 & -7.65 & 0 \\ 0 & 0 & 0 & -3.60 \end{bmatrix}.$$

This product multiplies a (4×4) matrix by another one, so multiplication is conformable and the product is also (4×4). The elements are inner-products of the corresponding row of Q and the corresponding column of B:

$$\begin{aligned}
&(1,1) \text{ element}: && (0.91 \times 10.47) + (0.35 \times 0) + (-0.38 \times 0) + (0.26 \times 0) && = 9.53, \\
&(1,2) \text{ element}: && (0.91 \times 0) + (0.35 \times -9.21) + (-0.38 \times 0) + (0.26 \times 0) && = -3.22, \\
&(1,3) \text{ element}: && (0.91 \times 0) + (0.35 \times 0) + (-0.38 \times -7.65) + (0.26 \times 0) && = 2.91, \\
&(1,4) \text{ element}: && (0.91 \times 0) + (0.35 \times 0) + (-0.38 \times 0) + (0.26 \times -3.60) && = -0.94, \\
&(2,1) \text{ element}: && (0.28 \times 10.47) + (-0.44 \times 0) + (0 \times 0) + (0.64 \times 0) && = 2.93, \\
&(2,2) \text{ element}: && (0.28 \times 0) + (-0.44 \times -9.21) + (0 \times 0) + (0.64 \times 0) && = 4.05, \\
&(2,3) \text{ element}: && (0.28 \times 0) + (-0.44 \times 0) + (0 \times -7.65) + (0.64 \times 0) && = 0, \\
&(2,4) \text{ element}: && (0.28 \times 0) + (-0.44 \times 0) + (0 \times 0) + (0.64 \times -3.60) && = -2.30, \\
&(3,1) \text{ element}: && (-0.27 \times 10.47) + (0.83 \times 0) + (-0.44 \times 0) + (0.03 \times 0) && = -2.83, \\
&(3,2) \text{ element}: && (-0.27 \times 0) + (0.83 \times -9.21) + (-0.44 \times 0) + (0.03 \times 0) && = -7.64, \\
&(3,3) \text{ element}: && (-0.27 \times 0) + (0.83 \times 0) + (-0.44 \times -7.65) + (0.03 \times 0) && = 3.37, \\
&(3,4) \text{ element}: && (-0.27 \times 0) + (0.83 \times 0) + (-0.44 \times 0) + (0.03 \times -3.60) && = -0.11, \\
&(4,1) \text{ element}: && (0.11 \times 10.47) + (0.03 \times 0) + (0.81 \times 0) + (-0.72 \times 0) && = 1.15, \\
&(4,2) \text{ element}: && (0.11 \times 0) + (0.03 \times -9.21) + (0.81 \times 0) + (-0.72 \times 0) && = -0.28, \\
&(4,3) \text{ element}: && (0.11 \times 0) + (0.03 \times 0) + (0.81 \times -7.65) + (-0.72 \times 0) && = -6.20, \\
&(4,4) \text{ element}: && (0.11 \times 0) + (0.03 \times 0) + (0.81 \times 0) + (-0.72 \times -3.60) && = 2.59.
\end{aligned}$$

Next we left-multiply this matrix by the inverse of Q:

$$(QB)Q^{-1} = \begin{bmatrix} 9.53 & -3.22 & 2.91 & -0.94 \\ 2.93 & 4.05 & 0 & -2.30 \\ -2.83 & -7.64 & 3.37 & -0.11 \\ 1.15 & -0.28 & -6.20 & 2.59 \end{bmatrix} \begin{bmatrix} 0.97 & -0.18 & -0.52 & 0.18 \\ 0.09 & 0.92 & 1.62 & 0.94 \\ -0.45 & 2.00 & 1.19 & 1.68 \\ -0.35 & 2.27 & 1.33 & 0.57 \end{bmatrix}.$$

Again, this product multiplies a (4×4) matrix by another one, so multiplication is conformable and the product is also (4×4). The elements are inner-products of the corresponding row of QB and the corresponding column of Q^{-1}:

$$
\begin{array}{lcl}
(1,1) \text{ element}: & (9.53 \times 0.97) + (-3.22 \times 0.09) + (2.91 \times -0.45) + (-0.94 \times -0.35) & = 8, \\
(1,2) \text{ element}: & (9.53 \times -0.18) + (-3.22 \times 0.92) + (2.91 \times 2.00) + (-0.94 \times 2.27) & = -1, \\
(1,3) \text{ element}: & (9.53 \times -0.52) + (-3.22 \times 1.62) + (2.91 \times 1.19) + (-0.94 \times 1.33) & = -8, \\
(1,4) \text{ element}: & (9.53 \times 0.18) + (-3.22 \times 0.94) + (2.91 \times 1.68) + (-0.94 \times 0.57) & = 3, \\
(2,1) \text{ element}: & (2.93 \times 0.97) + (4.05 \times 0.09) + (0 \times -0.45) + (-2.30 \times -0.35) & = 4, \\
(2,2) \text{ element}: & (2.93 \times -0.18) + (4.05 \times 0.92) + (0 \times 2.00) + (-2.30 \times 2.27) & = -2, \\
(2,3) \text{ element}: & (2.93 \times -0.52) + (4.05 \times 1.62) + (0 \times 1.19) + (-2.30 \times 1.33) & = 2, \\
(2,4) \text{ element}: & (2.93 \times 0.18) + (4.05 \times 0.94) + (0 \times 1.68) + (-2.30 \times 0.57) & = 3, \\
(3,1) \text{ element}: & (-2.83 \times 0.97) + (-7.64 \times 0.09) + (3.37 \times -0.45) + (-0.11 \times -0.35) & = -5, \\
(3,2) \text{ element}: & (-2.83 \times -0.18) + (-7.64 \times 0.92) + (3.37 \times 2.00) + (-0.11 \times 2.27) & = -0, \\
(3,3) \text{ element}: & (-2.83 \times -0.52) + (-7.64 \times 1.62) + (3.37 \times 1.19) + (-0.11 \times 1.33) & = -7, \\
(3,4) \text{ element}: & (-2.83 \times 0.18) + (-7.64 \times 0.94) + (3.37 \times 1.68) + (-0.11 \times 0.57) & = -2, \\
(4,1) \text{ element}: & (1.15 \times 0.97) + (-0.28 \times 0.09) + (-6.20 \times -0.45) + (2.59 \times -0.35) & = 3, \\
(4,2) \text{ element}: & (1.15 \times -0.18) + (-0.28 \times 0.92) + (-6.20 \times 2.00) + (2.59 \times 2.27) & = -7, \\
(4,3) \text{ element}: & (1.15 \times -0.52) + (-0.28 \times 1.62) + (-6.20 \times 1.19) + (2.59 \times 1.33) & = -5, \\
(4,4) \text{ element}: & (1.15 \times 0.18) + (-0.28 \times 0.94) + (-6.20 \times 1.68) + (2.59 \times 0.57) & = -9,
\end{array}
$$

so the total product is

$$
QBQ^{-1} = \begin{bmatrix} 8 & -1 & -8 & 3 \\ 4 & -2 & 2 & 3 \\ -5 & 0 & -7 & -2 \\ 3 & -7 & -5 & -9 \end{bmatrix},
$$

and we've demonstrated that $QBQ^{-1} = A$.

9. (a) The concept we are trying to measure is social capital. We characterize this concept to involve an individual's belief that other people can be trusted, and the individual's inclination to follow societal rules even when there will be no consequence for breaking them. An individual with a large amount of social capital will both trust others and will avoid making decisions that can harm others indirectly. Some of the concepts that we do not want to include in this characterization are: conforming to rules when sanctions for failing to do so are present, the size of an individual's social network, an individual's societal standing, prestige, or class. In this example, we are representing the concept of social capital with the survey question on trust and with the battery of questions regarding civic cooperation. In practice, we would probably want to add additional variables to represent the concept, but this is the representation for this simple example. We will measure the concept with principle components analysis.

   (b) In order to find the eigenvalues of the covariance matrix, we find the determinant of

$$
\begin{bmatrix} 6.32 - \lambda & 4.47 \\ 4.47 & 7.51 - \lambda \end{bmatrix},
$$

   and solve for λ such that the determinant is 0. The characteristic equation is

$$
(6.32 - \lambda)(7.51 - \lambda) - 4.47^2 = 0,
$$

$$(\lambda - 6.32)(\lambda - 7.51) - 19.98 = 0,$$
$$(\lambda^2 - 13.83\lambda + 47.46) - 19.98 = 0,$$
$$\lambda^2 - 13.83\lambda + 27.48 = 0,$$
$$\lambda = \frac{13.83 \pm \sqrt{(-13.83)^2 - 4(1)(27.48)}}{2},$$
$$= \frac{13.83 \pm \sqrt{191.27 - 109.92}}{2},$$
$$= \frac{13.83 \pm \sqrt{81.35}}{2},$$
$$= \frac{13.83 \pm 9.02}{2},$$

$$\lambda = \frac{13.83 - 9.02}{2} = 2.41, \qquad \lambda = \frac{13.83 + 9.02}{2} = 11.43.$$

To find the eigenvectors associated with $\lambda = 11.43$, we plug this eigenvalue into the above matrix,

$$\begin{bmatrix} 6.32 - 11.43 & 4.47 \\ 4.47 & 7.51 - 11.43 \end{bmatrix} = \begin{bmatrix} -5.11 & 4.47 \\ 4.47 & -3.92 \end{bmatrix},$$

and reduce as much as possible using elementary row operations. This task is trickier because we have no choice but to deal with the decimals. First we multiply the first row by 4.47/5.11=.875 and add it to the second row:

$$\begin{bmatrix} -5.11 & 4.47 \\ 0 & 0 \end{bmatrix}.$$

This reduction implies the following system of equations,

$$\begin{cases} -5.11x + 4.47y = 0, \\ y = y, \end{cases}$$

solving the top equation for x,

$$\begin{cases} x = .875y, \\ y = y, \end{cases}$$

so in matrix notation the eigenvectors associated with $\lambda = 11.43$ are

$$\begin{bmatrix} x \\ y \end{bmatrix} = y \begin{bmatrix} .875 \\ 1 \end{bmatrix}.$$

A particular eigenvector that arises when $y = 1$ is $\begin{bmatrix} .875 \\ 1 \end{bmatrix}$. To find the unit eigenvector, we compute the length of this vector and divide the vector by it's length. The length is

$$\left\| \begin{bmatrix} .875 \\ 1 \end{bmatrix} \right\| = \sqrt{.875^2 + 1^2} = 1.33.$$

Therefore the unit eigenvector associated with $\lambda = 11.43$ is

$$\frac{1}{1.33} \begin{bmatrix} .875 \\ 1 \end{bmatrix} = \begin{bmatrix} .658 \\ .752 \end{bmatrix}.$$

To find the eigenvectors associated with $\lambda = 2.41$, we plug this eigenvalue into the above matrix,

$$\begin{bmatrix} 6.32 - 2.41 & 4.47 \\ 4.47 & 7.51 - 2.41 \end{bmatrix} = \begin{bmatrix} 3.91 & 4.47 \\ 4.47 & 5.1 \end{bmatrix},$$

and reduce as much as possible using elementary row operations. First we multiply the first row by -4.47/3.91=-1.14 and add it to the second row:

$$\begin{bmatrix} 3.91 & 4.47 \\ 0 & 0 \end{bmatrix}.$$

This reduction implies the following system of equations,

$$\begin{cases} 3.91x + 4.47y = 0, \\ y = y, \end{cases}$$

solving the top equation for x,

$$\begin{cases} x = -1.14y, \\ y = y, \end{cases}$$

so in matrix notation the eigenvectors associated with $\lambda = 2.41$ are

$$\begin{bmatrix} x \\ y \end{bmatrix} = y \begin{bmatrix} -1.14 \\ 1 \end{bmatrix}.$$

A particular eigenvector that arises when $y = 1$ is $\begin{bmatrix} -1.14 \\ 1 \end{bmatrix}$. To find the unit eigenvector, we compute the length of this vector and divide the vector by it's length. The length is

$$\left\| \begin{bmatrix} -1.14 \\ 1 \end{bmatrix} \right\| = \sqrt{(-1.14)^2 + 1^2} = 1.52.$$

Therefore the unit eigenvector associated with $\lambda = 2.41$ is

$$\frac{1}{1.52} \begin{bmatrix} -1.14 \\ 1 \end{bmatrix} = \begin{bmatrix} -.75 \\ .658 \end{bmatrix}.$$

(c) The variance explained by the larger eigenvalue is

$$\frac{11.43}{11.43 + 2.41} = 82.6 \text{ percent}$$

(d) To create an index for social capital, we consider the largest eigenvalue and its unit eigenvector, and use the elements of this unit eigenvector to weight the observed variables that generated the covariance matrix. In part (b) we found that the largest eigenvalue is $\lambda = 11.43$ and that its unit eigenvector is

$$\begin{bmatrix} .658 \\ .752 \end{bmatrix}.$$

Therefore the equation for the social capital index is

$$\text{Social capital} = .658 \times \text{Trust} + .752 \times \text{Civic Cooperation}.$$

Including this index in the data gives us the following table:

Obs.	Trust	Civic Cooperation	Social Capital
1	4	2	.658(4)+.752(2)=4.1
2	7	10	.658(7)+.752(10)=12.1
3	3	6	.658(3)+.752(6)=6.5
4	8	9	.658(8)+.752(9)=12.0
5	9	7	.658(9)+.752(7)=11.2
6	2	3	.658(2)+.752(3)=3.6
7	9	6	.658(9)+.752(6)=10.4
8	5	3	.658(5)+.752(3)=5.5
9	6	4	.658(6)+.752(4)=7.0
10	8	8	.658(8)+.752(8)=11.3

10. This is a hard problem. Usually, researchers quickly turn to computers to process the tedious calculus involved with a problem like this. But it is worthwhile to struggle through one of these problems by hand, to get a strong sense of what the computer does and why. Be patient and careful as you work through the steps. Generally speaking, because we are rounding here to the third decimal place, our results by hand will be slightly different from the results we would get by using a computer, so don't be surprised to see small discrepancies if you compare these results to computer output.

    (a) The cross-tabulation can be written as

    $$M = \begin{bmatrix} 452 & 174 \\ 82 & 292 \end{bmatrix}.$$

    To calculate P, we find the sum of all elements in the cross-tab, $452 + 174 + 82 + 292 = 1000$, and divide every element of M by this sum,

    $$P = \begin{bmatrix} .452 & .174 \\ .082 & .292 \end{bmatrix}.$$

    (b) R is a vector that contains the row sums of P,

    $$R = \begin{bmatrix} 0.626 \\ 0.374 \end{bmatrix},$$

    and C is a vector that contains the column sums of P,

    $$C = \begin{bmatrix} 0.534 \\ 0.466 \end{bmatrix}.$$

    The matrix D_r is square with as many rows as R, has zeroes for the off-diagonal elements, and contains the square roots of the elements of R on its diagonal:

    $$D_r = \begin{bmatrix} \sqrt{0.626} & 0 \\ 0 & \sqrt{0.374} \end{bmatrix} = \begin{bmatrix} 0.719 & 0 \\ 0 & 0.611 \end{bmatrix}.$$

    Likewise, the matrix D_c is square with as many rows as C, has zeroes for the off-diagonal elements, and contains the square roots of the elements of C on its diagonal:

    $$D_c = \begin{bmatrix} \sqrt{0.534} & 0 \\ 0 & \sqrt{0.466} \end{bmatrix} = \begin{bmatrix} 0.731 & 0 \\ 0 & 0.682 \end{bmatrix}.$$

(c) To calculate the matrix $S = D_r(P - RC')D_c$ all we have to do is plug in these matrices and perform the calculations. Let's work step by step starting with RC':

$$RC' = \begin{bmatrix} 0.626 \\ 0.374 \end{bmatrix} \begin{bmatrix} 0.534 & 0.466 \end{bmatrix} = \begin{bmatrix} 0.334 & 0.292 \\ 0.200 & 0.174 \end{bmatrix}.$$

Next let's calculate $P - RC'$,

$$P - RC' = \begin{bmatrix} .452 & .174 \\ .082 & .292 \end{bmatrix} - \begin{bmatrix} 0.334 & 0.292 \\ 0.200 & 0.174 \end{bmatrix} = \begin{bmatrix} 0.118 & -0.118 \\ -0.118 & 0.118 \end{bmatrix}.$$

Next we calculate $D_r(P - Rc')$,

$$D_r(P - RC') = \begin{bmatrix} 0.719 & 0 \\ 0 & 0.611 \end{bmatrix} \begin{bmatrix} 0.118 & -0.118 \\ -0.118 & 0.118 \end{bmatrix} = \begin{bmatrix} 0.093 & -0.093 \\ -0.072 & 0.072 \end{bmatrix}.$$

Finally, we right-multiply this product by D_c,

$$S = D_r(P - RC')D_c = \begin{bmatrix} 0.093 & -0.093 \\ -0.072 & 0.072 \end{bmatrix} \begin{bmatrix} 0.731 & 0 \\ 0 & 0.682 \end{bmatrix} = \begin{bmatrix} 0.068 & -0.064 \\ -0.053 & 0.049 \end{bmatrix}.$$

(d) To find SS', we simply multiply

$$SS' = \begin{bmatrix} 0.068 & -0.064 \\ -0.053 & 0.049 \end{bmatrix} \begin{bmatrix} 0.068 & -0.053 \\ -0.064 & 0.049 \end{bmatrix} = \begin{bmatrix} 0.009 & -0.007 \\ -0.007 & 0.005 \end{bmatrix}.$$

Likewise, to find $S'S$, we multiply

$$S'S = \begin{bmatrix} 0.068 & -0.053 \\ -0.064 & 0.049 \end{bmatrix} \begin{bmatrix} 0.068 & -0.064 \\ -0.053 & 0.049 \end{bmatrix} = \begin{bmatrix} 0.007 & -0.007 \\ -0.007 & 0.007 \end{bmatrix}.$$

(e) First consider SS'. To find the eigenvalues of this matrix, we have to solve the characteristic equation

$$\left| \begin{bmatrix} 0.009 - \lambda & -0.007 \\ -0.007 & 0.005 - \lambda \end{bmatrix} \right| = 0,$$

$$(.009 - \lambda)(.005 - \lambda) - .007^2 = 0,$$

$$(\lambda - .009)(\lambda - .005) - .000049 = 0,$$

$$\lambda^2 - .014\lambda + .000045 - .000049 = 0,$$

$$\lambda^2 - .014\lambda - .000004 = 0.$$

The third term is close enough to 0 for us to round it to 0 (and anyway, if we wanted to perform arithmetic at the sixth decimal place we'd be using computers to do so), and so the eigenvalues of SS' are

$$\lambda^2 - .014\lambda = 0$$

$$\lambda(\lambda - .014) = 0,$$

$$\lambda = 0, \; \lambda = .014.$$

(f) In this step, we only have to rearrange the results we calculated in the previous step. U is the matrix of unit eigenvectors of SS', so

$$U = \begin{bmatrix} -0.791 & -0.612 \\ 0.612 & -0.791 \end{bmatrix}.$$

Σ is the diagonal matrix that contains the square roots of the eigenvalues on the diagonal:

$$\Sigma = \begin{bmatrix} \sqrt{0.014} & 0 \\ 0 & \sqrt{0} \end{bmatrix} = \begin{bmatrix} 0.118 & 0 \\ 0 & 0 \end{bmatrix}.$$

Finally, V is the matrix of unit eigenvectors of $S'S$, so

$$V = \begin{bmatrix} -0.731 & -0.683 \\ 0.683 & -0.731 \end{bmatrix}.$$

Recall that

$$S = \begin{bmatrix} 0.068 & -0.064 \\ -0.053 & 0.049 \end{bmatrix}.$$

If U, Σ, and V comprise the singular value decomposition of S, then

$$S = U\Sigma V'.$$

We have to compute $U\Sigma V'$ and show that it is equal to S. First, let's compute $U\Sigma$:

$$U\Sigma = \begin{bmatrix} -0.791 & -0.612 \\ 0.612 & -0.791 \end{bmatrix} \begin{bmatrix} 0.118 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} -0.093 & 0 \\ 0.072 & 0 \end{bmatrix}.$$

Next we compute $U\Sigma V'$:

$$U\Sigma V' = \begin{bmatrix} -0.093 & 0 \\ 0.072 & 0 \end{bmatrix} \begin{bmatrix} -0.731 & 0.683 \\ -0.683 & -0.731 \end{bmatrix} = \begin{bmatrix} 0.068 & -0.064 \\ -0.053 & 0.049 \end{bmatrix}.$$

So we have confirmed this singular value decomposition.

(g) To find the coordinates for the categories that comprise the rows of the cross-tabulation, we calculate $D_r U\Sigma$. In the previous step we calculated $U\Sigma$. To complete the product we multiply

$$D_r U\Sigma = \begin{bmatrix} 0.719 & 0 \\ 0 & 0.611 \end{bmatrix} \begin{bmatrix} -0.093 & 0 \\ 0.072 & 0 \end{bmatrix} = \begin{bmatrix} -0.074 & 0 \\ 0.044 & 0 \end{bmatrix}.$$

To find the coordinates for the categories that comprise the columns, we calculate $D_c V\Sigma$:

$$D_c V\Sigma = \begin{bmatrix} 0.731 & 0 \\ 0 & 0.682 \end{bmatrix} \begin{bmatrix} -0.731 & -0.683 \\ 0.683 & -0.731 \end{bmatrix} \begin{bmatrix} 0.118 & 0 \\ 0 & 0 \end{bmatrix}$$

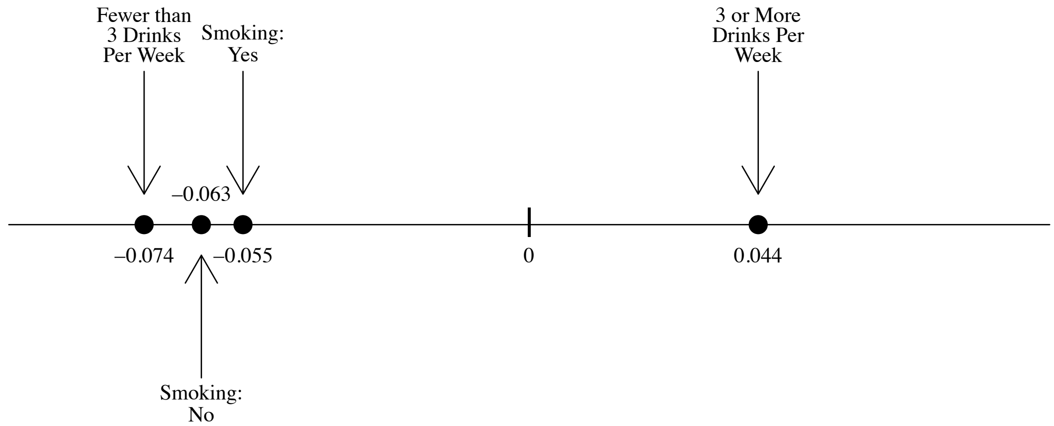
$$= \begin{bmatrix} -0.534 & 0.499 \\ -0.466 & -0.499 \end{bmatrix} \begin{bmatrix} 0.118 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} -0.063 & 0 \\ -0.055 & 0 \end{bmatrix}.$$

The coordinates for the categories that appear in the cross-tabulation are as follows:

Category	First latent variable	Second latent variable
Fewer than 3 drinks per week	-0.074	0
3 or more drinks per week	0.044	0
Regular smoker: No	-0.063	0
Regular smoker: Yes	-0.055	0

All of the coordinates for the second latent variable turn out to be zero – we do not have enough data to make a statement about a second dimension that underlies the data.

(h) The plot of these points (on a number line since there's no second dimension) is below:



The first latent variable indicates the similarity between categories. The standout is the "3 or more drinks per week" category.