Chapter 9: Matrix Inverses, Singularity, and Rank

1. There's really no way around tedious work to solve this problem. Either we have to take the inverse of a (4×4) matrix, or we can do slightly less work by showing that

$$\begin{bmatrix} 2 & 10 & 2 & -3 \\ 8 & -4 & 5 & -8 \\ 0 & 5 & -1 & 6 \\ -8 & 4 & -4 & 3 \end{bmatrix} \frac{1}{11} \begin{bmatrix} -44.5 & 138 & 99 & 125.5 \\ -4 & 15 & 11 & 14 \\ 100 & -309 & -220 & -284 \\ 20 & -64 & -44 & -59 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

and also that

$$\frac{1}{11} \begin{bmatrix} -44.5 & 138 & 99 & 125.5 \\ -4 & 15 & 11 & 14 \\ 100 & -309 & -220 & -284 \\ 20 & -64 & -44 & -59 \end{bmatrix} \begin{bmatrix} 2 & 10 & 2 & -3 \\ 8 & -4 & 5 & -8 \\ 0 & 5 & -1 & 6 \\ -8 & 4 & -4 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Let's try this latter approach. First consider the product

$$\begin{bmatrix} 2 & 10 & 2 & -3 \\ 8 & -4 & 5 & -8 \\ 0 & 5 & -1 & 6 \\ -8 & 4 & -4 & 3 \end{bmatrix} \frac{1}{11} \begin{bmatrix} -44.5 & 138 & 99 & 125.5 \\ -4 & 15 & 11 & 14 \\ 100 & -309 & -220 & -284 \\ 20 & -64 & -44 & -59 \end{bmatrix}.$$

For matrices, the order in which the matrices are multiplied matters, but for scalars the order still doesn't matter. So we can bring the factor $\frac{1}{11}$ to the front,

$$\frac{1}{11} \begin{bmatrix} 2 & 10 & 2 & -3 \\ 8 & -4 & 5 & -8 \\ 0 & 5 & -1 & 6 \\ -8 & 4 & -4 & 3 \end{bmatrix} \begin{bmatrix} -44.5 & 138 & 99 & 125.5 \\ -4 & 15 & 11 & 14 \\ 100 & -309 & -220 & -284 \\ 20 & -64 & -44 & -59 \end{bmatrix}.$$

This product multiplies a (4×4) matrix by another (4×4) matrix and results in a (4×4) matrix. We also have to divide every element of the product by 11. The elements of this product are

$$\begin{aligned}
(1,1) \text{ element} &= \tfrac{1}{11}\Big((2 \times -44.5) + (10 \times -4) + (2 \times 100) + (-3 \times 20)\Big) &= 1 \\
(1,2) \text{ element} &= \tfrac{1}{11}\Big((2 \times 138) + (10 \times 15) + (2 \times -309) + (-3 \times -64)\Big) &= 0 \\
(1,3) \text{ element} &= \tfrac{1}{11}\Big((2 \times 99) + (10 \times 11) + (2 \times -220) + (-3 \times -44)\Big) &= 0 \\
(1,4) \text{ element} &= \tfrac{1}{11}\Big((2 \times 125.5) + (10 \times 14) + (2 \times -284) + (-3 \times -59)\Big) &= 0 \\
(2,1) \text{ element} &= \tfrac{1}{11}\Big((8 \times -44,5) + (-4 \times -4) + (5 \times 100) + (-8 \times 20)\Big) &= 0 \\
(2,2) \text{ element} &= \tfrac{1}{11}\Big((8 \times 138) + (-4 \times 15) + (5 \times -309) + (-8 \times -64)\Big) &= 1 \\
(2,3) \text{ element} &= \tfrac{1}{11}\Big((8 \times 99) + (-4 \times 11) + (5 \times -220) + (-8 \times -44)\Big) &= 0 \\
(2,4) \text{ element} &= \tfrac{1}{11}\Big((8 \times 125.5) + (-4 \times 14) + (5 \times -284) + (-8 \times -59)\Big) &= 0 \\
(3,1) \text{ element} &= \tfrac{1}{11}\Big((0 \times -44.5) + (5 \times -4) + (-1 \times 100) + (6 \times 20)\Big) &= 0 \\
(3,2) \text{ element} &= \tfrac{1}{11}\Big((0 \times 138) + (5 \times 15) + (-1 \times -309) + (6 \times -64)\Big) &= 0 \\
(3,3) \text{ element} &= \tfrac{1}{11}\Big((0 \times 99) + (5 \times 11) + (-1 \times -220) + (6 \times -44)\Big) &= 1 \\
(3,4) \text{ element} &= \tfrac{1}{11}\Big((0 \times 125.5) + (5 \times 14) + (-1 \times -284) + (6 \times -59)\Big) &= 0 \\
(4,1) \text{ element} &= \tfrac{1}{11}\Big((-8 \times -44.5) + (4 \times -4) + (-4 \times 100) + (3 \times 20)\Big) &= 0 \\
(4,2) \text{ element} &= \tfrac{1}{11}\Big((-8 \times 138) + (4 \times 15) + (-4 \times -309) + (3 \times -64)\Big) &= 0 \\
(4,3) \text{ element} &= \tfrac{1}{11}\Big((-8 \times 99) + (4 \times 11) + (-4 \times -220) + (3 \times -44)\Big) &= 0 \\
(4,4) \text{ element} &= \tfrac{1}{11}\Big((-8 \times 125.5) + (4 \times 14) + (-4 \times -284) + (3 \times -59)\Big) &= 1
\end{aligned}$$

Therefore the product is equal to

$$\begin{bmatrix} 2 & 10 & 2 & -3 \\ 8 & -4 & 5 & -8 \\ 0 & 5 & -1 & 6 \\ -8 & 4 & -4 & 3 \end{bmatrix} \frac{1}{11} \begin{bmatrix} -44.5 & 138 & 99 & 125.5 \\ -4 & 15 & 11 & 14 \\ 100 & -309 & -220 & -284 \\ 20 & -64 & -44 & -59 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

and the first matrix the the left-inverse of the second. But to prove that these two matrices are full inverses of each other, we also have to compute

$$\frac{1}{11} \begin{bmatrix} -44.5 & 138 & 99 & 125.5 \\ -4 & 15 & 11 & 14 \\ 100 & -309 & -220 & -284 \\ 20 & -64 & -44 & -59 \end{bmatrix} \begin{bmatrix} 2 & 10 & 2 & -3 \\ 8 & -4 & 5 & -8 \\ 0 & 5 & -1 & 6 \\ -8 & 4 & -4 & 3 \end{bmatrix}.$$

Again, this product multiplies a (4×4) matrix by another (4×4) matrix and results in a (4×4) matrix. We also have to divide every element of the product by 11. The elements of this product are

$$\begin{aligned}
(1,1) \text{ element} &= \tfrac{1}{11}\Big((-44.5 \times 2) + (138 \times 8) + (99 \times 0) + (125.5 \times -8)\Big) &&= 1 \\
(1,2) \text{ element} &= \tfrac{1}{11}\Big((-44.5 \times 10) + (138 \times -4) + (99 \times 5) + (125.5 \times 4)\Big) &&= 0 \\
(1,3) \text{ element} &= \tfrac{1}{11}\Big((-44.5 \times 2) + (138 \times 5) + (99 \times -1) + (125.5 \times -4)\Big) &&= 0 \\
(1,4) \text{ element} &= \tfrac{1}{11}\Big((-44.5 \times -3) + (138 \times -8) + (99 \times 6) + (125.5 \times 3)\Big) &&= 0 \\
(2,1) \text{ element} &= \tfrac{1}{11}\Big((-4 \times 2) + (15 \times 8) + (11 \times 0) + (14 \times -8)\Big) &&= 0 \\
(2,2) \text{ element} &= \tfrac{1}{11}\Big((-4 \times 10) + (15 \times -4) + (11 \times 5) + (14 \times 4)\Big) &&= 1 \\
(2,3) \text{ element} &= \tfrac{1}{11}\Big((-4 \times 2) + (15 \times 5) + (11 \times -1) + (14 \times -4)\Big) &&= 0 \\
(2,4) \text{ element} &= \tfrac{1}{11}\Big((-4 \times -3) + (15 \times -8) + (11 \times 6) + (14 \times 3)\Big) &&= 0 \\
(3,1) \text{ element} &= \tfrac{1}{11}\Big((100 \times 2) + (-309 \times 8) + (-220 \times 0) + (-284 \times -8)\Big) &&= 0 \\
(3,2) \text{ element} &= \tfrac{1}{11}\Big((100 \times 10) + (-309 \times -4) + (-220 \times 5) + (-284 \times 4)\Big) &&= 0 \\
(3,3) \text{ element} &= \tfrac{1}{11}\Big((100 \times 2) + (-309 \times 5) + (-220 \times -1) + (-284 \times -4)\Big) &&= 1 \\
(3,4) \text{ element} &= \tfrac{1}{11}\Big((100 \times -3) + (-309 \times -8) + (-220 \times 6) + (-284 \times 3)\Big) &&= 0 \\
(4,1) \text{ element} &= \tfrac{1}{11}\Big((20 \times 2) + (-64 \times 8) + (-44 \times 0) + (-59 \times -8)\Big) &&= 0 \\
(4,2) \text{ element} &= \tfrac{1}{11}\Big((20 \times 10) + (-64 \times -4) + (-44 \times 5) + (-59 \times 4)\Big) &&= 0 \\
(4,3) \text{ element} &= \tfrac{1}{11}\Big((20 \times 2) + (-64 \times 5) + (-44 \times -1) + (-59 \times -4)\Big) &&= 0 \\
(4,4) \text{ element} &= \tfrac{1}{11}\Big((20 \times -3) + (-64 \times -8) + (-44 \times 6) + (-59 \times 3)\Big) &&= 1
\end{aligned}$$

So this product is also equal to the identity matrix,

$$\frac{1}{11}\begin{bmatrix} -44.5 & 138 & 99 & 125.5 \\ -4 & 15 & 11 & 14 \\ 100 & -309 & -220 & -284 \\ 20 & -64 & -44 & -59 \end{bmatrix}\begin{bmatrix} 2 & 10 & 2 & -3 \\ 8 & -4 & 5 & -8 \\ 0 & 5 & -1 & 6 \\ -8 & 4 & -4 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

and the two matrices are full inverses of each other.

2. We find the inverse of each of the following (2×2) matrices with elements denoted

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

by plugging their elements into the formula for a (2×2) inverse,

$$A^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

The inverse doesn't exist when $ad - bc = 0$.

(a) $A^{-1} = \dfrac{1}{(2\times 4)-(7\times 1)}\begin{bmatrix} 4 & -7 \\ -1 & 2 \end{bmatrix} = \dfrac{1}{1}\begin{bmatrix} 4 & -7 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 4 & -7 \\ -1 & 2 \end{bmatrix}.$

(b) $A^{-1} = \dfrac{1}{(4\times 5)-(5\times 3)}\begin{bmatrix} 5 & -3 \\ -5 & 4 \end{bmatrix} = \dfrac{1}{5}\begin{bmatrix} 5 & -3 \\ -5 & 4 \end{bmatrix} = \begin{bmatrix} 1 & -\frac{3}{5} \\ -1 & \frac{4}{5} \end{bmatrix}.$

(c) $A^{-1} = \dfrac{1}{(4\times -5)-(10\times -2)}\begin{bmatrix} -5 & -10 \\ 2 & 4 \end{bmatrix} = \dfrac{1}{\mathbf{0}}\begin{bmatrix} -5 & -10 \\ 2 & 4 \end{bmatrix}.$ This inverse does not exist.

(d) $A^{-1} = \dfrac{1}{(-10\times -1)-(2\times 8)}\begin{bmatrix} -1 & -2 \\ -8 & -10 \end{bmatrix} = -\dfrac{1}{6}\begin{bmatrix} -1 & -2 \\ -8 & -10 \end{bmatrix} = \begin{bmatrix} \frac{1}{6} & \frac{1}{3} \\ \frac{4}{3} & \frac{5}{3} \end{bmatrix}.$

(e) $A^{-1} = \dfrac{1}{(5\times 6)-(15\times 2)}\begin{bmatrix} 6 & -2 \\ -15 & 5 \end{bmatrix} = \dfrac{1}{\mathbf{0}}\begin{bmatrix} 6 & -2 \\ -15 & 5 \end{bmatrix}.$ This inverse does not exist.

(f) $A^{-1} = \dfrac{1}{(-2\times -6)-(6\times -7)}\begin{bmatrix} -6 & -6 \\ 7 & -2 \end{bmatrix} = \dfrac{1}{54}\begin{bmatrix} -6 & -6 \\ 7 & -2 \end{bmatrix} = \begin{bmatrix} -\frac{1}{9} & -\frac{1}{9} \\ \frac{7}{54} & -\frac{1}{27} \end{bmatrix}.$

(g) $A^{-1} = \dfrac{1}{(9\times 8)-(7\times 2)}\begin{bmatrix} 8 & -7 \\ -2 & 9 \end{bmatrix} = \dfrac{1}{58}\begin{bmatrix} 8 & -7 \\ -2 & 9 \end{bmatrix} = \begin{bmatrix} \frac{4}{29} & -\frac{7}{58} \\ -\frac{1}{29} & \frac{9}{58} \end{bmatrix}.$

(h) $A^{-1} = \dfrac{1}{(-5 \times 1) - (0 \times -7)} \begin{bmatrix} 1 & 0 \\ 7 & -5 \end{bmatrix} = -\dfrac{1}{5} \begin{bmatrix} 1 & 0 \\ 7 & -5 \end{bmatrix} = \begin{bmatrix} -\frac{1}{5} & 0 \\ -\frac{7}{5} & 1 \end{bmatrix}.$

3. (a) We first compute the minor elements for each position in the matrix by removing the indicated row and column and taking the determinant of the remaining (2×2) matrix. In this case, the minor elements are

$$\begin{aligned}
M_{11} &= \left| \begin{bmatrix} 4 & 6 \\ 0 & 5 \end{bmatrix} \right| &&= (4 \times 5) - (6 \times 0) &&= 20 \\
M_{12} &= \left| \begin{bmatrix} -3 & 6 \\ -9 & 5 \end{bmatrix} \right| &&= (-3 \times 5) - (6 \times -9) &&= 39 \\
M_{13} &= \left| \begin{bmatrix} -3 & 4 \\ -9 & 0 \end{bmatrix} \right| &&= (-3 \times 0) - (4 \times -9) &&= 36 \\
M_{21} &= \left| \begin{bmatrix} 6 & 6 \\ 0 & 5 \end{bmatrix} \right| &&= (6 \times 5) - (6 \times 0) &&= 30 \\
M_{22} &= \left| \begin{bmatrix} 3 & 6 \\ -9 & 5 \end{bmatrix} \right| &&= (3 \times 5) - (6 \times -9) &&= 69 \\
M_{23} &= \left| \begin{bmatrix} 3 & 6 \\ -9 & 0 \end{bmatrix} \right| &&= (3 \times 0) - (6 \times -9) &&= 54 \\
M_{31} &= \left| \begin{bmatrix} 6 & 6 \\ 4 & 6 \end{bmatrix} \right| &&= (6 \times 6) - (6 \times 4) &&= 12 \\
M_{32} &= \left| \begin{bmatrix} 3 & 6 \\ -3 & 6 \end{bmatrix} \right| &&= (3 \times 6) - (6 \times -3) &&= 36 \\
M_{33} &= \left| \begin{bmatrix} 3 & 6 \\ -3 & 4 \end{bmatrix} \right| &&= (3 \times 4) - (6 \times -3) &&= 30,
\end{aligned}$$

the cofactors are

$$C_{11} = (-1)^{1+1} M_{11} = 20, \quad C_{12} = (-1)^{1+2} M_{12} = -39, \quad C_{13} = (-1)^{3+1} M_{31} = 36,$$

$$C_{21} = (-1)^{2+1} M_{21} = -30, \quad C_{22} = (-1)^{2+2} M_{22} = 69, \quad C_{23} = (-1)^{3+2} M_{32} = -54,$$

$$C_{31} = (-1)^{3+1} M_{31} = 12, \quad C_{32} = (-1)^{3+2} M_{32} = -36, \quad C_{33} = (-1)^{3+3} M_{33} = 30,$$

and the cofactor matrix is therefore

$$C = \begin{bmatrix} 20 & -39 & 36 \\ -30 & 69 & -54 \\ 12 & -36 & 30 \end{bmatrix}.$$

Then the adjoint matrix is simply the transpose of the cofactor matrix,

$$C' = \begin{bmatrix} 20 & -30 & 12 \\ -39 & 69 & -36 \\ 36 & -54 & 30 \end{bmatrix}.$$

To find the determinant, we choose one row or column of the original matrix, multiply every element by its corresponding cofactor, and add. Using the first row gives us a determinant of

$$(3 \times 20) + (6 \times -39) + (6 \times 36) = 42.$$

The inverse is the adjoint matrix scalar multiplied by the reciprocal of the determinant,

$$\frac{1}{42}\begin{bmatrix} 20 & -30 & 12 \\ -39 & 69 & -36 \\ 36 & -54 & 30 \end{bmatrix} = \begin{bmatrix} 0.48 & -0.71 & 0.29 \\ -0.93 & 1.64 & -0.86 \\ 0.86 & -1.29 & 0.71 \end{bmatrix}.$$

(b) In this case, the minor elements are

$$\begin{aligned}
M_{11} &= \left|\begin{bmatrix} -8 & 1 \\ -2 & 3 \end{bmatrix}\right| &&= (-8 \times 3) - (1 \times -2) &&= -22 \\
M_{12} &= \left|\begin{bmatrix} 9 & 1 \\ 0 & 3 \end{bmatrix}\right| &&= (9 \times 3) - (1 \times 0) &&= 27 \\
M_{13} &= \left|\begin{bmatrix} 9 & -8 \\ 0 & -2 \end{bmatrix}\right| &&= (9 \times -2) - (-8 \times 0) &&= -18 \\
M_{21} &= \left|\begin{bmatrix} 10 & -5 \\ -2 & 3 \end{bmatrix}\right| &&= (10 \times 3) - (-5 \times -2) &&= 20 \\
M_{22} &= \left|\begin{bmatrix} -2 & -5 \\ 0 & 3 \end{bmatrix}\right| &&= (-2 \times 3) - (-5 \times 0) &&= -6 \\
M_{23} &= \left|\begin{bmatrix} -2 & 10 \\ 0 & -2 \end{bmatrix}\right| &&= (-2 \times -2) - (10 \times 0) &&= 4 \\
M_{31} &= \left|\begin{bmatrix} 10 & -5 \\ -8 & 1 \end{bmatrix}\right| &&= (10 \times 1) - (-5 \times -8) &&= -30 \\
M_{32} &= \left|\begin{bmatrix} -2 & -5 \\ 9 & 1 \end{bmatrix}\right| &&= (-2 \times 1) - (-5 \times 9) &&= 43 \\
M_{33} &= \left|\begin{bmatrix} -2 & 10 \\ 9 & -8 \end{bmatrix}\right| &&= (-2 \times -8) - (10 \times 9) &&= -74,
\end{aligned}$$

the cofactors are

$$C_{11} = (-1)^{1+1}M_{11} = -22, \quad C_{12} = (-1)^{1+2}M_{12} = -27, \quad C_{13} = (-1)^{3+1}M_{31} = -18,$$

$$C_{21} = (-1)^{2+1}M_{21} = -20, \quad C_{22} = (-1)^{2+2}M_{22} = -6, \quad C_{23} = (-1)^{3+2}M_{32} = -4,$$

$$C_{31} = (-1)^{3+1}M_{31} = -30, \quad C_{32} = (-1)^{3+2}M_{32} = -43, \quad C_{33} = (-1)^{3+3}M_{33} = -74,$$

and the cofactor matrix is therefore

$$C = \begin{bmatrix} -22 & -27 & -18 \\ -20 & -6 & -4 \\ -30 & -43 & -74 \end{bmatrix}.$$

Then the adjoint matrix is simply the transpose of the cofactor matrix,

$$C' = \begin{bmatrix} -22 & -20 & -30 \\ -27 & -6 & -43 \\ -18 & -4 & -74 \end{bmatrix}.$$

To find the determinant, we choose one row or column of the original matrix, multiply every element by its corresponding cofactor, and add. Using the first row gives us a determinant of

$$(-2 \times -22) + (10 \times -27) + (-5 \times -18) = -136.$$

The inverse is the adjoint matrix scalar multiplied by the reciprocal of the determinant,

$$-\frac{1}{136} \begin{bmatrix} -22 & -20 & -30 \\ -27 & -6 & -43 \\ -18 & -4 & -74 \end{bmatrix} = \begin{bmatrix} 0.16 & 0.15 & 0.22 \\ 0.20 & 0.04 & 0.32 \\ 0.13 & 0.03 & 0.54 \end{bmatrix}.$$

(c) In this case, the minor elements are

$$\begin{aligned}
M_{11} &= \left| \begin{bmatrix} 4 & 5 \\ 0 & -5 \end{bmatrix} \right| &&= (4 \times -5) - (5 \times 0) &&= -20 \\
M_{12} &= \left| \begin{bmatrix} -10 & 5 \\ 4 & -5 \end{bmatrix} \right| &&= (-10 \times -5) - (5 \times 4) &&= 30 \\
M_{13} &= \left| \begin{bmatrix} -10 & 4 \\ 4 & 0 \end{bmatrix} \right| &&= (-10 \times 0) - (4 \times 4) &&= -16 \\
M_{21} &= \left| \begin{bmatrix} 9 & 0 \\ 0 & -5 \end{bmatrix} \right| &&= (9 \times -5) - (0 \times 0) &&= -45 \\
M_{22} &= \left| \begin{bmatrix} 7 & 0 \\ 4 & -5 \end{bmatrix} \right| &&= (7 \times -5) - (0 \times 4) &&= -35 \\
M_{23} &= \left| \begin{bmatrix} 7 & 9 \\ 4 & 0 \end{bmatrix} \right| &&= (7 \times 0) - (9 \times 4) &&= -36 \\
M_{31} &= \left| \begin{bmatrix} 9 & 0 \\ 4 & 5 \end{bmatrix} \right| &&= (9 \times 5) - (0 \times 4) &&= 45 \\
M_{32} &= \left| \begin{bmatrix} 7 & 0 \\ -10 & 5 \end{bmatrix} \right| &&= (7 \times 5) - (0 \times -10) &&= 35 \\
M_{33} &= \left| \begin{bmatrix} 7 & 9 \\ -10 & 4 \end{bmatrix} \right| &&= (7 \times 4) - (9 \times -10) &&= 118,
\end{aligned}$$

the cofactors are

$$C_{11} = (-1)^{1+1} M_{11} = -20, \quad C_{12} = (-1)^{1+2} M_{12} = -30, \quad C_{13} = (-1)^{3+1} M_{31} = -16,$$

$$C_{21} = (-1)^{2+1} M_{21} = 45, \quad C_{22} = (-1)^{2+2} M_{22} = -35, \quad C_{23} = (-1)^{3+2} M_{32} = -36,$$

$$C_{31} = (-1)^{3+1} M_{31} = 45, \quad C_{32} = (-1)^{3+2} M_{32} = -35, \quad C_{33} = (-1)^{3+3} M_{33} = 118,$$

and the cofactor matrix is therefore

$$C = \begin{bmatrix} -20 & -30 & -16 \\ 45 & -35 & -36 \\ 45 & -35 & 118 \end{bmatrix}.$$

Then the adjoint matrix is simply the transpose of the cofactor matrix,

$$C' = \begin{bmatrix} -20 & 45 & 45 \\ -30 & -35 & -35 \\ -16 & -36 & 118 \end{bmatrix}.$$

To find the determinant, we choose one row or column of the original matrix, multiply every element by its corresponding cofactor, and add. Using the first row gives us a determinant of

$$(7 \times -20) + (9 \times -30) + (0 \times -16) = -410.$$

The inverse is the adjoint matrix scalar multiplied by the reciprocal of the determinant,

$$-\frac{1}{410}\begin{bmatrix} -20 & 45 & 45 \\ -30 & -35 & -35 \\ -16 & -36 & 118 \end{bmatrix} = \begin{bmatrix} 0.05 & -0.11 & -0.11 \\ 0.07 & 0.09 & 0.09 \\ 0.04 & -0.09 & -0.29 \end{bmatrix}.$$

(d) In this case, the minor elements are

$$\begin{aligned}
M_{11} &= \left|\begin{bmatrix} -7 & 2 \\ 7 & 2 \end{bmatrix}\right| &&= (-7 \times 2) - (2 \times 7) &&= -28 \\
M_{12} &= \left|\begin{bmatrix} 3 & 2 \\ 7 & 2 \end{bmatrix}\right| &&= (3 \times 2) - (2 \times 7) &&= -8 \\
M_{13} &= \left|\begin{bmatrix} 3 & -7 \\ 7 & 7 \end{bmatrix}\right| &&= (3 \times 7) - (-7 \times 7) &&= 70 \\
M_{21} &= \left|\begin{bmatrix} 1 & -1 \\ 7 & 2 \end{bmatrix}\right| &&= (1 \times 2) - (-1 \times 7) &&= 9 \\
M_{22} &= \left|\begin{bmatrix} 9 & -1 \\ 7 & 2 \end{bmatrix}\right| &&= (9 \times 2) - (-1 \times 7) &&= 25 \\
M_{23} &= \left|\begin{bmatrix} 9 & 1 \\ 7 & 7 \end{bmatrix}\right| &&= (9 \times 7) - (1 \times 7) &&= 56 \\
M_{31} &= \left|\begin{bmatrix} 1 & -1 \\ -7 & 2 \end{bmatrix}\right| &&= (1 \times 2) - (-1 \times -7) &&= -5 \\
M_{32} &= \left|\begin{bmatrix} 9 & -1 \\ 3 & 2 \end{bmatrix}\right| &&= (9 \times 2) - (-1 \times 3) &&= 21 \\
M_{33} &= \left|\begin{bmatrix} 9 & 1 \\ 3 & -7 \end{bmatrix}\right| &&= (9 \times -7) - (1 \times 3) &&= -66,
\end{aligned}$$

the cofactors are

$$C_{11} = (-1)^{1+1}M_{11} = -28, \qquad C_{12} = (-1)^{1+2}M_{12} = 8, \qquad C_{13} = (-1)^{3+1}M_{31} = 70,$$

$$C_{21} = (-1)^{2+1}M_{21} = -9, \qquad C_{22} = (-1)^{2+2}M_{22} = 25, \quad C_{23} = (-1)^{3+2}M_{32} = -56,$$

$$C_{31} = (-1)^{3+1}M_{31} = -5, \quad C_{32} = (-1)^{3+2}M_{32} = -21, \quad C_{33} = (-1)^{3+3}M_{33} = -66,$$

and the cofactor matrix is therefore

$$C = \begin{bmatrix} -28 & 8 & 70 \\ -9 & 25 & -56 \\ -5 & -21 & -66 \end{bmatrix}.$$

Then the adjoint matrix is simply the transpose of the cofactor matrix,

$$C' = \begin{bmatrix} -28 & -9 & -5 \\ 8 & 25 & -21 \\ 70 & -56 & -66 \end{bmatrix}.$$

To find the determinant, we choose one row or column of the original matrix, multiply every element by its corresponding cofactor, and add. Using the first row gives us a determinant of

$$(9 \times -28) + (1 \times 8) + (-1 \times 70) = -314.$$

The inverse is the adjoint matrix scalar multiplied by the reciprocal of the determinant,

$$-\frac{1}{314}\begin{bmatrix} -28 & -9 & -5 \\ 8 & 25 & -21 \\ 70 & -56 & -66 \end{bmatrix} = \begin{bmatrix} 0.09 & 0.03 & 0.02 \\ -0.03 & -0.08 & 0.07 \\ -0.22 & 0.18 & 0.21 \end{bmatrix}.$$

(e) In this case, the minor elements are

$$
\begin{aligned}
M_{11} &= \left|\begin{bmatrix} -2 & 4 \\ 2 & 6 \end{bmatrix}\right| &= (-2 \times 6) - (4 \times 2) &= -20 \\
M_{12} &= \left|\begin{bmatrix} -3 & 4 \\ 2 & 6 \end{bmatrix}\right| &= (-3 \times 6) - (4 \times 2) &= -26 \\
M_{13} &= \left|\begin{bmatrix} -3 & -2 \\ 2 & 2 \end{bmatrix}\right| &= (-3 \times 2) - (-2 \times 2) &= -2 \\
M_{21} &= \left|\begin{bmatrix} 6 & 10 \\ 2 & 6 \end{bmatrix}\right| &= (6 \times 6) - (10 \times 2) &= 16 \\
M_{22} &= \left|\begin{bmatrix} -5 & 10 \\ 2 & 6 \end{bmatrix}\right| &= (-5 \times 6) - (10 \times 2) &= -50 \\
M_{23} &= \left|\begin{bmatrix} -5 & 6 \\ 2 & 2 \end{bmatrix}\right| &= (-5 \times 2) - (6 \times 2) &= -22 \\
M_{31} &= \left|\begin{bmatrix} 6 & 10 \\ -2 & 4 \end{bmatrix}\right| &= (6 \times 4) - (10 \times -2) &= 44 \\
M_{32} &= \left|\begin{bmatrix} -5 & 10 \\ -3 & 4 \end{bmatrix}\right| &= (-5 \times 4) - (10 \times -3) &= 10 \\
M_{33} &= \left|\begin{bmatrix} -5 & 6 \\ -3 & -2 \end{bmatrix}\right| &= (-5 \times -2) - (6 \times -3) &= 28,
\end{aligned}
$$

the cofactors are

$$
\begin{aligned}
C_{11} = (-1)^{1+1} M_{11} = -20, \quad C_{12} = (-1)^{1+2} M_{12} = 26, \quad C_{13} = (-1)^{3+1} M_{31} = -2, \\
C_{21} = (-1)^{2+1} M_{21} = -16, \quad C_{22} = (-1)^{2+2} M_{22} = -50, \quad C_{23} = (-1)^{3+2} M_{32} = 22, \\
C_{31} = (-1)^{3+1} M_{31} = 44, \quad C_{32} = (-1)^{3+2} M_{32} = -10, \quad C_{33} = (-1)^{3+3} M_{33} = 28,
\end{aligned}
$$

and the cofactor matrix is therefore

$$
C = \begin{bmatrix} -20 & 26 & -2 \\ -16 & -50 & 22 \\ 44 & -10 & 28 \end{bmatrix}.
$$

Then the adjoint matrix is simply the transpose of the cofactor matrix,

$$
C' = \begin{bmatrix} -20 & -16 & 44 \\ 26 & -50 & -10 \\ -2 & 22 & 28 \end{bmatrix}.
$$

To find the determinant, we choose one row or column of the original matrix, multiply every element by its corresponding cofactor, and add. Using the first row gives us a determinant of

$$
(-5 \times -20) + (6 \times 26) + (10 \times -2) = 236.
$$

The inverse is the adjoint matrix scalar multiplied by the reciprocal of the determinant,

$$
-\frac{1}{236} \begin{bmatrix} -20 & -16 & 44 \\ 26 & -50 & -10 \\ -2 & 22 & 28 \end{bmatrix} = \begin{bmatrix} -0.08 & -0.07 & 0.19 \\ 0.11 & -0.21 & -0.04 \\ -0.01 & 0.09 & 0.12 \end{bmatrix}.
$$

(f) In this case, the minor elements are

$$
\begin{array}{lclcl}
M_{11} & = & \left|\begin{bmatrix} -3 & -5 \\ 0 & 4 \end{bmatrix}\right| & = & (-3 \times 4) - (-5 \times 0) & = & -12 \\
M_{12} & = & \left|\begin{bmatrix} 0 & -5 \\ -9 & 4 \end{bmatrix}\right| & = & (0 \times 4) - (-5 \times -9) & = & -45 \\
M_{13} & = & \left|\begin{bmatrix} 0 & -3 \\ -9 & 0 \end{bmatrix}\right| & = & (0 \times 0) - (-3 \times -9) & = & -27 \\
M_{21} & = & \left|\begin{bmatrix} 8 & 6 \\ 0 & 4 \end{bmatrix}\right| & = & (8 \times 4) - (6 \times 0) & = & 32 \\
M_{22} & = & \left|\begin{bmatrix} 3 & 6 \\ -9 & 4 \end{bmatrix}\right| & = & (3 \times 4) - (6 \times -9) & = & 66 \\
M_{23} & = & \left|\begin{bmatrix} 3 & 8 \\ -9 & 0 \end{bmatrix}\right| & = & (3 \times 0) - (8 \times -9) & = & 72 \\
M_{31} & = & \left|\begin{bmatrix} 8 & 6 \\ -3 & -5 \end{bmatrix}\right| & = & (8 \times -5) - (6 \times -3) & = & -22 \\
M_{32} & = & \left|\begin{bmatrix} 3 & 6 \\ 0 & -5 \end{bmatrix}\right| & = & (3 \times -5) - (6 \times 0) & = & -15 \\
M_{33} & = & \left|\begin{bmatrix} 3 & 8 \\ 0 & -3 \end{bmatrix}\right| & = & (3 \times -3) - (8 \times 0) & = & -9
\end{array}
$$

the cofactors are

$$
\begin{array}{rrr}
C_{11} = (-1)^{1+1}M_{11} = -12, & C_{12} = (-1)^{1+2}M_{12} = 45, & C_{13} = (-1)^{3+1}M_{31} = -27, \\
C_{21} = (-1)^{2+1}M_{21} = 32, & C_{22} = (-1)^{2+2}M_{22} = 66, & C_{23} = (-1)^{3+2}M_{32} = -72, \\
C_{31} = (-1)^{3+1}M_{31} = -22, & C_{32} = (-1)^{3+2}M_{32} = 15, & C_{33} = (-1)^{3+3}M_{33} = -9,
\end{array}
$$

and the cofactor matrix is therefore

$$
C = \begin{bmatrix} -12 & 45 & -27 \\ 32 & 66 & -72 \\ -22 & 15 & -9 \end{bmatrix}.
$$

Then the adjoint matrix is simply the transpose of the cofactor matrix,

$$
C' = \begin{bmatrix} -12 & 32 & -22 \\ 45 & 66 & -72 \\ -27 & -72 & -9 \end{bmatrix}.
$$

To find the determinant, we choose one row or column of the original matrix, multiply every element by its corresponding cofactor, and add. Using the first row gives us a determinant of

$$
(3 \times -12) + (8 \times 45) + (6 \times -27) = 162.
$$

The inverse is the adjoint matrix scalar multiplied by the reciprocal of the determinant,

$$
-\frac{1}{162}\begin{bmatrix} -12 & 32 & -22 \\ 45 & 66 & -72 \\ -27 & -72 & -9 \end{bmatrix} = \begin{bmatrix} -0.07 & -0.20 & -0.14 \\ 0.28 & 0.41 & 0.09 \\ -0.17 & -0.44 & -0.06 \end{bmatrix}.
$$

4. (a) To perform Sarrus' rule, write the matrix twice, side by side. Then for every element in the first row of the left matrix, draw an oval moving down and to the right that contains three elements. Multiply the elements in each oval together. In this case, these products are

$$\begin{aligned}(6 \times 4 \times 6) &= 162,\\ (-9 \times 10 \times -9) &= 144,\\ (-8 \times -5 \times 8) &= 320.\end{aligned}$$

For every element in the first row of the right matrix, draw an oval moving down and to the left that contains three elements. In this case, these products are

$$\begin{aligned}(6 \times 10 \times 8) &= 480,\\ (-9 \times -5 \times 6) &= 270,\\ (-8 \times 4 \times -9) &= 288.\end{aligned}$$

Finally, we add the three products from the ovals moving down and to the right, and subtract the products from the ovals moving down and to the left. The determinant is:

$$(144 + 810 + 320) - (480 + 270 + 288) = 236.$$

(b) We write the matrix twice, side by side. Then for every element in the first row of the left matrix, we draw an oval moving down and to the right that contains three elements and multiply the elements in each oval together. In this case, these products are

$$\begin{aligned}(-9 \times -3 \times 6) &= 162,\\ (8 \times 4 \times 15) &= 480,\\ (-9 \times 1 \times -7) &= 63.\end{aligned}$$

For every element in the first row of the right matrix, we draw an oval moving down and to the left that contains three elements. In this case, these products are

$$\begin{aligned}(-9 \times 4 \times -7) &= 252,\\ (8 \times 1 \times 6) &= 48,\\ (-9 \times -3 \times 15) &= 405.\end{aligned}$$

Finally, we add the three products from the ovals moving down and to the right, and subtract the products from the ovals moving down and to the left. The determinant is:

$$(162 + 480 + 63) - (252 + 48 + 405) = 0.$$

(c) We write the matrix twice, side by side. Then for every element in the first row of the left matrix, we draw an oval moving down and to the right that contains three elements and multiply the elements in each oval together. In this case, these products are

$$\begin{aligned}(3 \times -6 \times -8) &= 144,\\ (0 \times -6 \times -2) &= 0,\\ (-2 \times 5 \times -4) &= 40.\end{aligned}$$

For every element in the first row of the right matrix, we draw an oval moving down and to the left that contains three elements. In this case, these products are

$$\begin{aligned}(3 \times -6 \times -4) &= 72,\\ (0 \times 5 \times -8) &= 0,\\ (-2 \times -6 \times -2) &= -24.\end{aligned}$$

Finally, we add the three products from the ovals moving down and to the right, and subtract the products from the ovals moving down and to the left. The determinant is:

$$(144 + 0 + 40) - (72 + 0 + -24) = 136.$$

(d) We write the matrix twice, side by side. Then for every element in the first row of the left matrix, we draw an oval moving down and to the right that contains three elements and multiply the elements in each oval together. In this case, these products are

$$\begin{aligned}(-8 \times -8 \times 10) &= 640,\\ (-3 \times 3 \times 0) &= 0,\\ (-8 \times -4 \times -1) &= -32.\end{aligned}$$

For every element in the first row of the right matrix, we draw an oval moving down and to the left that contains three elements. In this case, these products are

$$\begin{aligned}(-8 \times -8 \times 10) &= 640,\\ (-3 \times 3 \times 0) &= 0,\\ (-8 \times -4 \times -1) &= -32.\end{aligned}$$

Finally, we add the three products from the ovals moving down and to the right, and subtract the products from the ovals moving down and to the left. The determinant is:

$$(640 + 0 + -32) - (24 + 120 + 0) = 464.$$

5. A (4×4) matrix is much more unwieldy than a (3×3) matrix. Fortunately, we only have to find the determinant, not the full inverse matrix. The steps we will take are as follows:

- We have to choose one row or column to work with, so let's work with the first row.
- We will calculate the minor elements that correspond to each of the four elements on the first row. To find a minor element, we remove the indicated row and column from the (4×4) matrix and find the determinant of the remaining (3×3) matrix. Let's use Sarrus' rule to find those determinants.

- We will find the cofactors.
- We will multiply each element on the first row of the original matrix by its cofactor and add. The sum is the determinant of the (4×4) matrix.

First, we find the four minor elements for the first row. For the (1,1) minor element, we remove the first row and first column of the (4×4) matrix and take the determinant of what remains,

$$M_{11} = \left| \begin{bmatrix} 3 & 0 & 1 \\ -6 & -2 & -4 \\ -6 & 6 & 3 \end{bmatrix} \right|,$$

by applying Sarrus' rule,

$$\begin{aligned} (3 \times -2 \times 3) &= -18, \\ (0 \times -4 \times -6) &= 0, \\ (1 \times -6 \times 6) &= -36, \\ (3 \times -4 \times 6) &= -72, \\ (0 \times -6 \times 3) &= 0, \\ (1 \times -2 \times -6) &= 12, \end{aligned}$$

$$M_{11} = (-18 + 0 + -36) - (-72 + 0 + 12) = 6.$$

For the (1,2) minor element, we remove the first row and second column of the (4×4) matrix and take the determinant of what remains,

$$M_{12} = \left| \begin{bmatrix} -5 & 0 & 1 \\ 7 & -2 & -4 \\ 8 & 6 & 3 \end{bmatrix} \right|,$$

by applying Sarrus' rule,

$$\begin{aligned} (-5 \times -2 \times 3) &= 30, \\ (0 \times -4 \times 8) &= 0, \\ (1 \times 7 \times 6) &= 42, \\ (-5 \times -4 \times 6) &= 120, \\ (0 \times 7 \times 3) &= 0, \\ (1 \times -2 \times 8) &= -16, \end{aligned}$$

$$M_{12} = (30 + 0 + 42) - (120 + 0 + -16) = -32.$$

For the (1,3) minor element, we remove the first row and third column of the (4×4) matrix and take the determinant of what remains,

$$M_{13} = \left| \begin{bmatrix} -5 & 3 & 1 \\ 7 & -6 & -4 \\ 8 & -6 & 3 \end{bmatrix} \right|,$$

by applying Sarrus' rule,

$$\begin{aligned} (-5 \times -6 \times 3) &= 90, \\ (3 \times -4 \times 8) &= -96, \\ (1 \times 7 \times -6) &= -42, \\ (-5 \times -4 \times -6) &= -120, \\ (3 \times 7 \times 3) &= 63, \\ (1 \times -6 \times 8) &= -48, \end{aligned}$$

$$M_{13} = (90 + -96 + -42) - (-120 + 63 + -48) = 57.$$

Finally, for the (1,4) minor element, we remove the first row and third column of the (4×4) matrix and take the determinant of what remains,

$$M_{14} = \left| \begin{bmatrix} -5 & 3 & 0 \\ 7 & -6 & -2 \\ 8 & -6 & 6 \end{bmatrix} \right|,$$

by applying Sarrus' rule,

$$\begin{aligned} (-5 \times -6 \times 6) &= 180, \\ (3 \times -2 \times 8) &= -48, \\ (0 \times 7 \times -6) &= 0, \\ (-5 \times -2 \times -6) &= -60, \\ (3 \times 7 \times 6) &= 126, \\ (0 \times -6 \times 8) &= 0, \end{aligned}$$

$$M_{14} = (180 + -48 + 0) - (-60 + 126 + 0) = 66.$$

The cofactors are

$$C_{11} = (-1)^{1+1} M_{11} = 6, \quad C_{12} = (-1)^{1+2} M_{12} = 32,$$

$$C_{13} = (-1)^{1+3} M_{13} = 57, \quad C_{14} = (-1)^{1+4} M_{14} = -66.$$

And the last step is to multiply the elements on the first row of the original matrix by their cofactors and add. The sum is the determinant of the (4×4) matrix:

$$(-3 \times 6) + (-3 \times 32) + (-1 \times 57) + (-9 \times -66) = 423.$$

6. (a) As we saw in the previous problem, in order to find the determinant of a (4×4) matrix, it is necessary to take the find the determinant of 4 (3×3) matrices. Putting aside the time it takes to identify these matrices, calculate the cofactors, and add the products of the elements on a row and the cofactors, the time it should take a student to take this determinant will be *at least* 4 minutes.

   For a (5×5) matrix, the student can expand along the first row which requires the student to find 5 determinants of (4×4) matrices, each of which takes at least 4 minutes to find. So the calculation will take at least $5 \times 4 = 20$ minutes.

   For a (6×6) matrix, the student can expand along the first row which requires the student to find 6 determinants of (5×5) matrices, each of which takes at least 20 minutes to find. So the calculation will take at least $6 \times 20 = 120$ minutes (2 hours).

   And for a (7×7) matrix, the student can expand along the first row which requires the student to find 7 determinants of (6×6) matrices, each of which takes at least 120 minutes to find. So the calculation will take at least $7 \times 120 = 840$ minutes (14 hours!).

   In contrast, the computer can take a (4×4) determinant in about $4(.1) = .4$ seconds, a (5×5) determinant in about $5(.4) = 2$ seconds, a (6×6) determinant in about $6(2) = 12$ seconds, and a (7×7) determinant in about $7(12) = 84$ seconds.

   (b) The quote by Michael Sand refers to the fact that amount of tedious work involved in performing these matrix arithmetic computations increases exponentially with the size of the matrices. For anything larger than (4×4) the computation is a real project, and anything with 6 or 7 or more dimensions becomes quickly untenable. These computations are not difficult, they are tedious. And humans have severe constraints on the amount of tedious work they are able to perform quickly and accurately. In contrast, a computer is designed to excel at monotonous tasks. This advantage of computers is the primary reason why modern applied statistics in the social sciences relies so heavily on computers.

7. A matrix is singular only when its determinant is zero. Our strategy here will be to find the (2×2) determinants in terms of λ and then to find the value(s) of λ that make the determinant equal 0.

   (a)

$$\left|\begin{bmatrix} 8-\lambda & 7 \\ 7 & 8-\lambda \end{bmatrix}\right| = 0,$$

$$(8-\lambda)^2 - 49 = 0,$$

$$(64 - 16\lambda + \lambda^2) - 49 = 0,$$

$$\lambda^2 - 16\lambda + 15 = 0,$$

$$(\lambda - 15)(\lambda - 1) = 0,$$

$$\lambda = 1, \lambda = 15.$$

   (b)

$$\left|\begin{bmatrix} 9-\lambda & -9 \\ 7 & -7-\lambda \end{bmatrix}\right| = 0,$$

$$(-9-\lambda)(-7-\lambda) - 63 = 0,$$

$$(63 + 16\lambda + \lambda^2) - 63 = 0,$$

$$\lambda^2 + 16\lambda = 0,$$

$$\lambda(\lambda + 16) = 0,$$

$$\lambda = 0, \lambda = -16.$$

(c)

$$\left| \begin{bmatrix} 2-\lambda & -8 \\ -10 & 10-\lambda \end{bmatrix} \right| = 0,$$

$$(2-\lambda)(10-\lambda) - 80 = 0,$$

$$(20 - 12\lambda + \lambda^2) - 80 = 0,$$

$$\lambda^2 - 12\lambda - 60 = 0,$$

$$\lambda = \frac{12 \pm \sqrt{144 - 4(1)(-60)}}{2},$$

$$\lambda = \frac{12 \pm \sqrt{144 + 240}}{2},$$

$$\lambda = \frac{12 \pm \sqrt{384}}{2},$$

$$\lambda = \frac{12 \pm 8\sqrt{6}}{2},$$

$$\lambda = 6 \pm 4\sqrt{6},$$

$$\lambda = 6 - 4\sqrt{6} = -3.80,$$
$$\lambda = 6 + 4\sqrt{6} = 15.80.$$

8. We can reduce the matrix as much as possible using the elementary row operations. If the matrix reduces to an identity matrix, then its rank is full and equal to the number of columns. If it contains a column of all zeroes then it's rank is equal to the number of columns that do not consist entirely of zeroes.

   (a) With the goal of reducing this matrix to the identity matrix, let's start by interchanging the first and second rows,

   $$\begin{bmatrix} 1 & 7 & -1 \\ 3 & -3 & 0 \\ -4 & 4 & 6 \end{bmatrix},$$

   multiplying the first row by -3 and adding it to the second row,

   $$\begin{bmatrix} 1 & 7 & -1 \\ 0 & -24 & -3 \\ -4 & 4 & 6 \end{bmatrix},$$

   and multiplying the first row by 4 and adding it to the third row,

   $$\begin{bmatrix} 1 & 7 & -1 \\ 0 & -24 & -3 \\ 0 & 32 & 10 \end{bmatrix}.$$

Next, we divide the second row by -3 and the third row by 2,

$$\begin{bmatrix} 1 & 7 & -1 \\ 0 & 8 & 1 \\ 0 & 16 & 5 \end{bmatrix},$$

multiply the second row by -2 and add it to the third row,

$$\begin{bmatrix} 1 & 7 & -1 \\ 0 & 8 & 1 \\ 0 & 0 & 3 \end{bmatrix},$$

and divide the third row by 3,

$$\begin{bmatrix} 1 & 7 & -1 \\ 0 & 8 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

We now add the third row to the first, and we multiply the third row by -1 and add it to the second,

$$\begin{bmatrix} 1 & 7 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

divide the second row by 8,

$$\begin{bmatrix} 1 & 7 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

and add -7 times the second row to the first row,

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Since the matrix breaks down to an identity matrix, it is of full rank – the rank is 3.

(b) Let's begin by interchanging the first and second rows,

$$\begin{bmatrix} 1 & 6 & 0 & 2 \\ -2 & 4 & -1 & 3 \\ 3 & 5 & -2 & -1 \\ 4 & 14 & -5 & 1 \end{bmatrix}.$$

We add 2 times the first row to the second, -3 times the first row to the third, and -4 times the first row to the fourth,

$$\begin{bmatrix} 1 & 6 & 0 & 2 \\ 0 & 16 & -1 & 7 \\ 0 & -13 & -2 & -7 \\ 0 & -10 & -5 & -7 \end{bmatrix}.$$

To avoid fractions in the next step, we multiply the first, second and third rows by 10, and interchange the second and fourth rows

$$\begin{bmatrix} 10 & 60 & 0 & 20 \\ 0 & -10 & -5 & -7 \\ 0 & 160 & -10 & 70 \\ 0 & -130 & -20 & -70 \end{bmatrix},$$

we then add 16 times the second row to the third,

$$\begin{bmatrix} 10 & 60 & 0 & 20 \\ 0 & -10 & -5 & -7 \\ 0 & 0 & -90 & -42 \\ 0 & -130 & -20 & -70 \end{bmatrix},$$

and we add -13 times the second row to the fourth row,

$$\begin{bmatrix} 10 & 60 & 0 & 20 \\ 0 & -10 & -5 & -7 \\ 0 & 0 & -90 & -42 \\ 0 & 0 & 45 & 21 \end{bmatrix}.$$

We divide the third row by 2,

$$\begin{bmatrix} 10 & 60 & 0 & 20 \\ 0 & -10 & -5 & -7 \\ 0 & 0 & -45 & -21 \\ 0 & 0 & 45 & 21 \end{bmatrix},$$

and add it to the fourth row

$$\begin{bmatrix} 10 & 60 & 0 & 20 \\ 0 & -10 & -5 & -7 \\ 0 & 0 & -45 & -21 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

We next divide the third row by -3,

$$\begin{bmatrix} 10 & 60 & 0 & 20 \\ 0 & -10 & -5 & -7 \\ 0 & 0 & 15 & 7 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

add 6 times the second row to the first,

$$\begin{bmatrix} 10 & 0 & -30 & -22 \\ 0 & -10 & -5 & -7 \\ 0 & 0 & 15 & 7 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

and add 2 times the third row to the first,

$$\begin{bmatrix} 10 & 0 & 0 & -8 \\ 0 & -10 & -5 & -7 \\ 0 & 0 & 15 & 7 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

To continue avoiding fractions, we multiply the second row by 3,

$$\begin{bmatrix} 10 & 0 & 0 & -8 \\ 0 & -30 & -15 & -21 \\ 0 & 0 & 15 & 7 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

and add the third row to the second,

$$\begin{bmatrix} 10 & 0 & 0 & -8 \\ 0 & -30 & 0 & -14 \\ 0 & 0 & 15 & 7 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Finally, we divide the first row by 10, the second row by -30, and the third row by 15,

$$\begin{bmatrix} 1 & 0 & 0 & -\frac{4}{5} \\ 0 & 1 & 0 & \frac{7}{15} \\ 0 & 0 & 1 & \frac{7}{15} \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Since the first three rows and columns contain an identity matrix, we've reduced the matrix as much as we can. There are 3 rows that do not consist entirely of zeroes, so the rank of the matrix is 3. Since this matrix is not of full column rank, we also know that it is singular, that there is a linear dependency in its rows and columns, that its determinant is zero, and that it has no inverse.

(c) Let's start by interchanging the first and third rows,

$$\begin{bmatrix} 1 & 4 & 11 & 1 \\ -4 & 6 & 8 & 1 \\ 5 & -2 & 3 & 0 \\ 9 & -8 & -5 & -1 \end{bmatrix},$$

then multiplying the first row by 4 and adding it to the second row, multiplying it by -5 and adding it to the third row, and multiplying it by -9 and adding it to the third row,

$$\begin{bmatrix} 1 & 4 & 11 & 1 \\ 0 & 22 & 49 & 5 \\ 0 & -22 & -52 & -5 \\ 0 & -44 & -104 & -10 \end{bmatrix}.$$

Next we add the second row to the third, and add 2 times the second row to the fourth row,

$$\begin{bmatrix} 1 & 4 & 11 & 1 \\ 0 & 22 & 49 & 5 \\ 0 & 0 & -3 & 0 \\ 0 & 0 & -6 & 0 \end{bmatrix}.$$

We add 2 times the third row to the fourth row,

$$\begin{bmatrix} 1 & 4 & 11 & 1 \\ 0 & 22 & 49 & 5 \\ 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

divide the third row by -3,

$$\begin{bmatrix} 1 & 4 & 11 & 1 \\ 0 & 22 & 49 & 5 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

and add -49 times the third row to the second row and -11 times the third row to the first row,

$$\begin{bmatrix} 1 & 4 & 0 & 1 \\ 0 & 22 & 0 & 5 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

We multiply the first row by 11,

$$\begin{bmatrix} 11 & 44 & 0 & 11 \\ 0 & 22 & 0 & 5 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

and add -2 times the second row to the first,

$$\begin{bmatrix} 11 & 0 & 0 & 1 \\ 0 & 22 & 0 & 5 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Finally we divide the first row by 11 and the second row by 22,

$$\begin{bmatrix} 1 & 0 & 0 & \frac{1}{11} \\ 0 & 1 & 0 & \frac{5}{22} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Since the first three rows and columns contain an identity matrix, we've reduced the matrix as much as we can. There are 3 rows that do not consist entirely of zeroes, so the rank of the matrix is 3. Since this matrix is not of full column rank, we also know that it is singular, that there is a linear dependency in its rows and columns, that its determinant is zero, and that it has no inverse.

(d) Note that this matrix,

$$\begin{bmatrix} 1 & -5 & 8 \\ 2 & -3 & 4 \\ 6 & 0 & 3 \\ 9 & 1 & 3 \\ -4 & 4 & 0 \end{bmatrix},$$

is not square. Since it has more rows than columns, it cannot be full row rank but it might be full column rank. To reduce, start by multiplying the first row by -2 and adding it to the third row, then multiply it by -6 and add it to the third row, then multiply it by -9 and add it to the fourth row, and multiply it by 4 and add it to the fifth row,

$$\begin{bmatrix} 1 & -5 & 8 \\ 0 & 7 & -12 \\ 0 & 30 & -45 \\ 0 & -44 & -69 \\ 0 & -16 & 32 \end{bmatrix}.$$

Next, interchange the second and fifth rows,

$$\begin{bmatrix} 1 & -5 & 8 \\ 0 & -16 & 32 \\ 0 & 30 & -45 \\ 0 & -44 & -69 \\ 0 & 7 & -12 \end{bmatrix},$$

divide the second row by -16, divide the third row by 15, and divide the fourth row by -3,

$$\begin{bmatrix} 1 & -5 & 8 \\ 0 & 1 & -2 \\ 0 & 2 & -3 \\ 0 & 18 & -23 \\ 0 & 7 & -12 \end{bmatrix}.$$

Multiply the second row by-2 and add it to the third row, then multiply the second row by -18 and add it to the fourth row, and multiply the second row by -7 and add it to the fifth row,

$$\begin{bmatrix} 1 & -5 & 8 \\ 0 & 1 & -2 \\ 0 & 0 & -1 \\ 0 & 0 & 13 \\ 0 & 0 & 2 \end{bmatrix},$$

and multiply the third row by -1,

$$\begin{bmatrix} 1 & -5 & 8 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \\ 0 & 0 & 13 \\ 0 & 0 & 2 \end{bmatrix}.$$

Multiply the third row by -8, 2, -13, and -2 and add it to the first, second, fourth, and fifth rows respectively,

$$\begin{bmatrix} 1 & -5 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

and add 5 times the second row to the first row,

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Since we've reduced the matrix so that there is an identity matrix in the first three rows and columns, we've reduced as much as we can. There are three rows that do not consist entirely of zeroes, so the rank is 3, and the matrix is of full column rank.

9. First we calculate the matrix product

$$X'X = \begin{bmatrix} 6 & 4 & 7 & 2 & 6 & 1 & 3 & 3 & 2 & 1 \\ 6 & 3 & 5 & 2 & 6 & 1 & 4 & 3 & 2 & 1 \\ 12 & 7 & 12 & 4 & 12 & 2 & 7 & 6 & 4 & 2 \end{bmatrix} \begin{bmatrix} 6 & 6 & 12 \\ 4 & 3 & 7 \\ 7 & 5 & 12 \\ 2 & 2 & 4 \\ 6 & 6 & 12 \\ 1 & 1 & 2 \\ 3 & 4 & 7 \\ 3 & 3 & 6 \\ 2 & 2 & 4 \\ 1 & 1 & 2 \end{bmatrix},$$

This product multiplies a (3×10) matrix by a (10×3) matrix, which is conformable and results in a (3×3) matrix. As we saw in chapter 8 exercise 4, any matrix that is left-multiplied by its transpose will be symmetric. Therefore it will only be necessary to calculate the 6 unique elements. The (1,1) element is

$$(6 \times 6) + (4 \times 4) + (7 \times 7) + (2 \times 2) + (6 \times 6) + (1 \times 1) + (3 \times 3) + (3 \times 3) + (2 \times 2) + (1 \times 1) = 165.$$

The (1,2) and (2,1) elements are

$$(6 \times 6) + (4 \times 3) + (7 \times 5) + (2 \times 2) + (6 \times 6) + (1 \times 1) + (3 \times 4) + (3 \times 3) + (2 \times 2) + (1 \times 1) = 150.$$

The (1,3) and (3,1) elements are

$$(6 \times 12) + (4 \times 7) + (7 \times 12) + (2 \times 4) + (6 \times 12) + (1 \times 2) + (3 \times 7) + (3 \times 6) + (2 \times 4) + (1 \times 2) = 315.$$

The (2,2) element is

$$(6 \times 6) + (3 \times 3) + (5 \times 5) + (2 \times 2) + (6 \times 6) + (1 \times 1) + (4 \times 4) + (3 \times 3) + (2 \times 2) + (1 \times 1) = 141.$$

The (2,3) and (3,2) elements are

$$(6 \times 12) + (3 \times 7) + (5 \times 12) + (2 \times 4) + (6 \times 12) + (1 \times 2) + (4 \times 7) + (3 \times 6) + (2 \times 4) + (1 \times 2) = 291.$$

And the (3,3) element is

$$(12 \times 12) + (7 \times 7) + (12 \times 12) + (4 \times 4) + (12 \times 12) + (2 \times 2) + (7 \times 7) + (6 \times 6) + (4 \times 4) + (2 \times 2) = 606.$$

The entire product is

$$X'X = \begin{bmatrix} 165 & 150 & 315 \\ 150 & 141 & 291 \\ 315 & 291 & 606 \end{bmatrix}.$$

To find the determinant, we apply Sarrus' rule. We write the matrix twice, adjacent to itself. Then we draw six ovals. Three ovals start in the first row of the left matrix, and move diagonally down and to the right, encompassing three elements. Three ovals start in the first row of the right matrix, and move diagonally down and to the left, also encompassing three elements. We multiply the elements within each oval together, add the products for the ovals that move down and right, and subtract the products of the ovals that move down and left.

The products from the ovals that move down and right are

$$\begin{aligned} (165 \times 141 \times 606) &= 14,098,590, \\ (150 \times 291 \times 315) &= 13,749,750, \\ (315 \times 150 \times 291) &= 13,749,750. \end{aligned}$$

The products from the ovals that move down and left are

$$\begin{aligned} (165 \times 291 \times 291) &= 13,972,365, \\ (150 \times 150 \times 606) &= 13,635,000, \\ (315 \times 141 \times 315) &= 13,990,725. \end{aligned}$$

The determinant is

$$|X'X| = (14,098,590 + 13,749,750 + 13,749,750) - (13,972,365 + 13,635,000 + 13,990,725) = 0.$$

10. (a) We are trying to find the product

$$X'Y = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 2 & 1 & 2 & 0 & 1 & 3 & 11 & 0 & 5 & 0 \\ 5 & 1 & 0 & 2 & 0 & 0 & 0 & 0 & 12 & 2 \end{bmatrix} \begin{bmatrix} 10 \\ 7 \\ 2 \\ 8 \\ 10 \\ 6 \\ 2 \\ 10 \\ 8 \\ 5 \end{bmatrix}$$

$$= \begin{bmatrix} (1\times 10)+(1\times 7)+(1\times 2)+(1\times 8)+(1\times 10)+(1\times 6)+(1\times 2)+(1\times 10)+(1\times 8)+(1\times 5) \\ (2\times 10)+(1\times 7)+(2\times 2)+(0\times 8)+(1\times 10)+(3\times 6)+(11\times 2)+(0\times 10)+(5\times 8)+(0\times 5) \\ (5\times 10)+(1\times 7)+(0\times 2)+(2\times 8)+(0\times 10)+(0\times 6)+(0\times 2)+(0\times 10)+(12\times 8)+(2\times 5) \end{bmatrix}$$

$$= \begin{bmatrix} 68 \\ 121 \\ 179 \end{bmatrix}.$$

(b) We are computing the product

$$X'X = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 2 & 1 & 2 & 0 & 1 & 3 & 11 & 0 & 5 & 0 \\ 5 & 1 & 0 & 2 & 0 & 0 & 0 & 0 & 12 & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 & 5 \\ 1 & 1 & 1 \\ 1 & 2 & 0 \\ 1 & 0 & 2 \\ 1 & 1 & 0 \\ 1 & 3 & 0 \\ 1 & 11 & 0 \\ 1 & 0 & 0 \\ 1 & 5 & 12 \\ 1 & 0 & 2 \end{bmatrix},$$

which will be a symmetric (3×3) matrix. We have to compute the (1,1), (1,2), (1,3), (2,2), (2,3), and (3,3) elements.

The (1,1) element is

$$(1\times 1)+(1\times 1)+(1\times 1)+(1\times 1)+(1\times 1)+(1\times 1)+(1\times 1)+(1\times 1)+(1\times 1)+(1\times 1) = 10.$$

The (1,2) and (2,1) elements are

$$(1\times 2)+(1\times 1)+(1\times 2)+(1\times 0)+(1\times 1)+(1\times 3)+(1\times 11)+(1\times 0)+(1\times 5)+(1\times 0) = 25.$$

The (1,3) and (3,1) elements are

$$(1\times 5)+(1\times 1)+(1\times 0)+(1\times 2)+(1\times 0)+(1\times 0)+(1\times 0)+(1\times 0)+(1\times 12)+(1\times 2) = 22.$$

The (2,2) element is

$$(2\times 2)+(1\times 1)+(2\times 2)+(0\times 0)+(1\times 1)+(3\times 3)+(11\times 11)+(0\times 0)+(5\times 5)+(0\times 0) = 165.$$

The (2,3) and (3,2) elements are

$$(2\times 5)+(1\times 1)+(2\times 0)+(0\times 2)+(1\times 0)+(3\times 0)+(11\times 0)+(0\times 0)+(5\times 12)+(0\times 2) = 71.$$

Finally, the (3,3) element is

$$(5 \times 5) + (1 \times 1) + (0 \times 0) + (2 \times 2) + (0 \times 0) + (0 \times 0) + (0 \times 0) + (0 \times 0) + (12 \times 12) + (2 \times 2) = 178.$$

So the entire matrix is

$$X'X = \begin{bmatrix} 10 & 25 & 22 \\ 25 & 165 & 71 \\ 22 & 71 & 178 \end{bmatrix}.$$

(c) In order to find the inverse of $X'X$, we will apply the formula

$$(X'X)^{-1} = \frac{1}{|X'X|} \text{adj}(X'X),$$

which involves finding the adjoint matrix, then the determinant. To find the adjoint matrix, we first determine the minor elements of $X'X$, which for the (i, j)th element involves removing the ith row and the jth column of $X'X$ and calculating the determinant of the remaining (2×2) matrix. In general, the determinant of a (2×2) matrix is given by

$$\left| \begin{bmatrix} a & b \\ c & d \end{bmatrix} \right| = ad - bc.$$

The matrix of minor elements is symmetric because $X'X$ is symmetric, and therefore removing the ith row and jth column is the same as removing the jth row and ith column. As a result, we only need to calculate 6 minor elements.

To find the $(1, 1)$ minor element, we remove the first row and first column from $X'X$ and find the determinant:

$$M_{11} = \left| \begin{bmatrix} 165 & 71 \\ 71 & 178 \end{bmatrix} \right| = (165 \times 178) - (71 \times 71) = 24,329.$$

To find the $(1, 2)$ and $(2, 1)$ minor elements:

$$M_{12} = M_{21} = \left| \begin{bmatrix} 25 & 71 \\ 22 & 178 \end{bmatrix} \right| = (25 \times 178) - (22 \times 71) = 2,888.$$

To find the $(1, 3)$ and $(3, 1)$ minor elements:

$$M_{13} = M_{31} = \left| \begin{bmatrix} 25 & 165 \\ 22 & 71 \end{bmatrix} \right| = (25 \times 71) - (22 \times 165) = -1,855.$$

To find the $(2, 2)$ minor element:

$$M_{22} = \left| \begin{bmatrix} 10 & 22 \\ 22 & 178 \end{bmatrix} \right| = (10 \times 178) - (22 \times 22) = 1,296$$

To find the $(2, 3)$ and $(3, 2)$ minor elements:

$$M_{23} = M_{32} = \left| \begin{bmatrix} 10 & 25 \\ 22 & 71 \end{bmatrix} \right| = (10 \times 71) - (22 \times 25) = 160$$

Finally, to find the $(3, 3)$ minor element:

$$M_{33} = \left| \begin{bmatrix} 10 & 25 \\ 25 & 165 \end{bmatrix} \right| = (10 \times 165) - (25 \times 25) = 1,025.$$

The entire matrix of minor elements is therefore

$$M = \begin{bmatrix} 24,329 & 2,888 & -1,855 \\ 2,888 & 1,296 & 160 \\ -1,855 & 160 & 1,025 \end{bmatrix}.$$

Next we find the cofactor matrix by multiplying the elements of the matrix of minor elements by -1 if the row and column numbers of the element add to an odd number. The cofactor matrix is

$$C = \begin{bmatrix} 24,329 & -2,888 & -1,855 \\ -2,888 & 1,296 & -160 \\ -1,855 & -160 & 1,025 \end{bmatrix}.$$

Next, the adjoint matrix is the transpose of the cofactor matrix. But since $X'X$, M, and C are all symmetric in this case, the adjoint matrix is equal to the cofactor matrix:

$$\text{adj}(X'X) = \begin{bmatrix} 24,329 & -2,888 & -1,855 \\ -2,888 & 1,296 & -160 \\ -1,855 & -160 & 1,025 \end{bmatrix}.$$

To find the determinant of the (3×3) matrix, we choose one row or column of the matrix, multiply each element by its corresponding cofactor element, and sum these products. Let's choose the first row of $X'X$:

$$\begin{bmatrix} 10 & 25 & 22 \end{bmatrix}$$

The corresponding cofactor elements are

$$\begin{bmatrix} 24,329 & -2,888 & -1,855 \end{bmatrix}.$$

Multiplying the corresponding elements and adding gives us:

$$|X'X| = (10 \times 24,329) + (25 \times -2,888) + (22 \times -1,855) = 130,280.$$

Finally, plugging the adjoint matrix and determinant into

$$(X'X)^{-1} = \frac{1}{|X'X|}\text{adj}(X'X),$$

gives us

$$(X'X)^{-1} = \frac{1}{130,280}\begin{bmatrix} 24,329 & -2,888 & -1,855 \\ -2,888 & 1,296 & -160 \\ -1,855 & -160 & 1,025 \end{bmatrix} = \begin{bmatrix} 0.187 & -0.022 & -0.014 \\ -0.022 & 0.010 & -0.001 \\ -0.014 & -0.001 & 0.008 \end{bmatrix}.$$

(d) Having already gone through all of the work to calculate $(X'X)^{-1}$ and $X'Y$, all we have left to do is multiply these two matrices:

$$\hat{\beta} = (X'X)^{-1}(X'Y) = \begin{bmatrix} 0.187 & -0.022 & -0.014 \\ -0.022 & 0.010 & -0.001 \\ -0.014 & -0.001 & 0.008 \end{bmatrix}\begin{bmatrix} 68 \\ 121 \\ 179 \end{bmatrix}.$$

The product multiplies a (3×3) matrix by a (3×1) matrix, so multiplication is possible and the product is the following (3×1) vector:

$$\hat{\beta} = (X'X)^{-1}(X'Y) = \begin{bmatrix} (0.187 \times 68) + (-0.022 \times 121) + (-0.014 \times 179) \\ (-0.022 \times 68) + (0.010 \times 121) + (-0.001 \times 179) \\ (-0.014 \times 68) + (-0.001 \times 121) + (0.008 \times 179) \end{bmatrix} = \begin{bmatrix} 7.47 \\ -0.52 \\ 0.29 \end{bmatrix}.$$

So the OLS estimate for α is 7.47, for β_1 is -0.52, and for β_2 is 0.29. A computer can perform these calculations in a fraction of a second. But now, you have mastered the mathematics that the computer uses. Hopefully, this process demystifies the process of OLS regression, and you can look at a computer as a useful tool, but one no more intelligent than a handheld calculator.