Chapter 1: Algebra Review

1. First I consider the parentheses, calculating division first,

$$5 \times (3 - 2) + 6 \div 2 - 7 \times 0,$$

then subtraction,

$$5 \times 1 + 6 \div 2 - 7 \times 0.$$

Now that the parentheses are solved, I perform multiplication and division next,

$$5 + 6 \div 2 - 7 \times 0,$$

$$5 + 3 - 7 \times 0,$$

$$5 + 3 - 0,$$

which evaluates to 8.

2. (a) Here is one way to proceed:

$$\begin{aligned} &19800 \\ &= 100 \times 198 \\ &= (10 \times 10) \times 198 \\ &= (2 \times 5) \times (2 \times 5) \times 198 \\ &= 2 \times 2 \times 5 \times 5 \times (2 \times 99) \\ &= 2 \times 2 \times 2 \times 5 \times 5 \times (9 \times 11) \\ &= 2 \times 2 \times 2 \times 5 \times 5 \times (3 \times 3) \times 11 \\ &= 2 \times 2 \times 2 \times 3 \times 3 \times 5 \times 5 \times 11. \end{aligned}$$

(b) From the previous problem, we know that

$$\sqrt{19800} = \sqrt{2 \times 2 \times 2 \times 3 \times 3 \times 5 \times 5 \times 11}.$$

We use the "jailbreak method" to remove pairs of prime factors from the square root. There is a pair of 2s, of 3s, and of 5s to remove, leaving a 2 and the 11 inside:

$$= (2 \times 3 \times 5)\sqrt{2 \times 11}.$$

$$= 30\sqrt{22}.$$

3. (a) The lowest common denominator for 8 and 12 is 24. So we rewrite both fractions to have a denominator of 24. For the first fraction, we multiply the top and bottom by 3, and for the second fraction we multiply the top and bottom by 2:

$$\frac{5}{8} + \frac{1}{12} = \frac{15}{24} + \frac{2}{24} = \frac{17}{24}.$$

(b) We can cross-cancel a factor of 3 from both 3 and 21, and we can cross-cancel a factor of 5 from both 5 and 10. These cancelations leave us with

$$\frac{1}{1} \times \frac{2}{7} = \frac{2}{7}.$$

(c) We take the reciprocal of the denominator and write the problem as multiplication:

$$\frac{2}{7} \times \frac{8}{3}.$$

Neither 2 and 3 nor 7 and 8 share common factors. So we just multiply the numerators and denominators together:

$$\frac{2 \times 8}{7 \times 3} = \frac{16}{21}.$$

4. (a) First we can distribute the exponent 3 to each variable inside the parentheses,

$$x^3 y^3 x^2,$$

and we can rewrite the expression as

$$x^3 x^2 y^3.$$

Finally, we add the exponents for the factors that have a base of x:

$$x^5 y^3.$$

(b) First we take the reciprocal of the bottom fraction and write the expression as a multiplication problem:

$$\frac{w^3 z^4}{(w+1)(z-3)} \times \frac{(w-2)(z-3)}{(wz)^3} = \frac{w^3 z^4 (w-2)(z-3)}{(w+1)(z-3)(wz)^3}.$$

The factor $(z-3)$ is on the top and bottom and cancels out:

$$\frac{w^3 z^4 (w-2)}{(w+1)(wz)^3}.$$

We distribute the exponent of 3 to each variable inside (wz):

$$\frac{w^3 z^4 (w-2)}{(w+1) w^3 z^3}.$$

The factor w^3 is on the top and bottom and cancels out:

$$\frac{z^4 (w-2)}{(w+1) z^3},$$

and we subtract the exponents on z, leaving

$$\frac{z(w-2)}{(w+1)}.$$

(c) We can take a number of steps in any order. Here's one solution. First, distribute the exponent 3 to the x^2 and to y:

$$\frac{\frac{(x^2)^3}{y^3}x^{-2}}{xy^2}$$

For the $(x^2)^3$ term, we multiply the exponents together to get x^6:

$$\frac{\frac{x^6}{y^3}x^{-2}}{xy^2}.$$

Next we can remove all the fractions by multiplying the exponents of terms in denominators by -1. So the y^3 becomes y^{-3}, x becomes x^{-1}, and y^2 becomes y^{-2}:

$$x^6x^{-2}x^{-1}y^{-3}y^{-2}.$$

We can add the exponents of factors that share the same base, leaving

$$x^3y^{-5}.$$

Finally we can move y^{-5} back to the denominator in order to write it with a positive exponent:

$$\frac{x^3}{y^5}.$$

(d) The "jailbreak" method works for cube-roots too, except three repetitions of the same prime factor are needed in order to bring one factor outside the root. The prime factorization of 189 is

$$3 \times 63$$
$$3 \times (3 \times 21)$$
$$3 \times 3 \times 3 \times 7.$$

So the problem becomes

$${}_3\sqrt{3 \times 3 \times 3 \times 7}.$$

The 3 repetitions of the prime factor 3 allow one 3 to leave the root, leaving only 7 inside the root:

$$3_3\sqrt{7}.$$

(e) We can rewrite the problem as

$$x^{\frac{1}{3}}x^{\frac{1}{5}},$$

and then we can add the exponents:

$$x^{\frac{1}{3}+\frac{1}{5}}.$$

In order to add two fractions with different denominators, we find the lowest common denominator for 3 and 5, which is 15. Rewriting each fraction the expression becomes

$$x^{\frac{5}{15}+\frac{3}{15}} = x^{\frac{8}{15}}.$$

We're not quite done yet. Fractional exponents are somewhat messy, so we can rewrite the exponent as

$$(x^8)^{\frac{1}{15}},$$

which means that the expression becomes

$${}_{15}\sqrt{x^8}.$$

5. (a)

$$2 \times 2 \times 2 \times \ldots \times 2 \text{ (128 times)}$$
$$= 2^{128}$$

Stata tells me this number is approximately 3.403e+38, which is

$$340,300,000,000,000,000,000,000,000,000,000,000,000$$

(b)

$$\frac{340,300,000,000,000,000,000,000,000,000,000,000,000 \text{ keys}}{1,000,000,000,000,000 \text{ keys per second}}$$
$$= 340,300,000,000,000,000,000,000 \text{ seconds}$$
$$\text{Divide by 60:} \quad = 5,672,000,000,000,000,000,000 \text{ minutes}$$
$$\text{Divide by 60:} \quad = 94,530,000,000,000,000,000 \text{ hours}$$
$$\text{Divide by 24:} \quad = 3,939,000,000,000,000,000 \text{ days}$$
$$\text{Divide by 365:} \quad = 10,790,000,000,000,000 \text{ years,}$$

which is approximately 1000 times longer than the age of the universe.

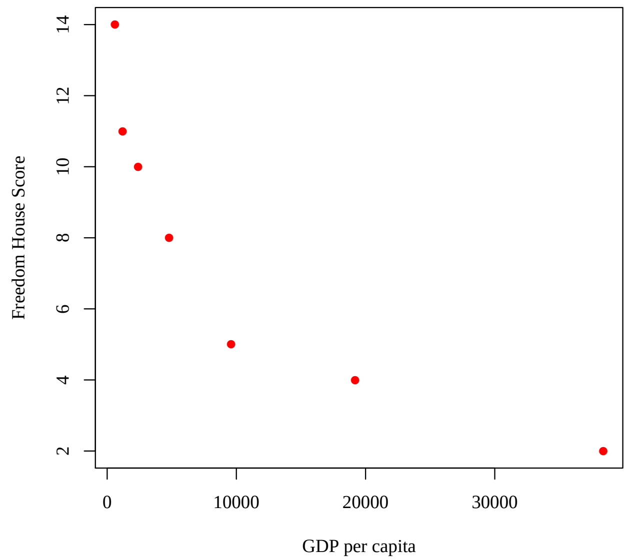
(c)

$$\sqrt{2^{128}} = (2^{128})^{.5} = 2^{128 \times .5} = 2^{64}$$
$$\text{Approximately } 18,450,000,000,000,000,000$$

(d)

$$\frac{2^{64} \text{ keys}}{1,000,000,000,000,000 \text{ keys per second}} \approx 18447 \text{ seconds}$$
$$\text{Divide by 60:} \quad \approx 307 \text{ minutes}$$
$$\text{Divide by 60:} \quad \approx 5.1 \text{ hours}$$

6. (a) The function is an exponential function, and no straight line fits the data very well.



(b) These values can be represented in terms of g as follows:

Guinea-Bissau: $\log_2(1200) = \log_2(600 \times 2) = \log_2(600) + \log_2(2) = g + 1.$

Kyrgyzstan: $\log_2(2400) = \log_2(600 \times 4) = \log_2(600) + \log_2(2^2) = \log_2(600) + 2\log_2(2) = g + 2.$

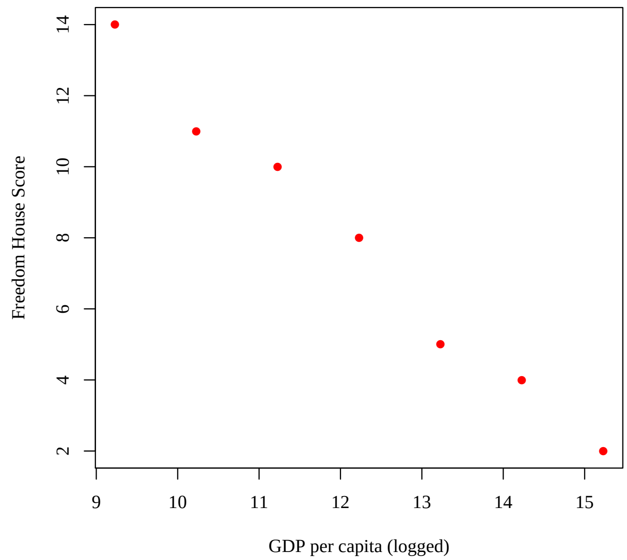
Honduras: $\log_2(4800) = \log_2(600 \times 8) = \log_2(600) + \log_2(2^3) = \log_2(600) + 3\log_2(2) = g + 3.$

Jamaica: $\log_2(9600) = \log_2(600 \times 16) = \log_2(600) + \log_2(2^4) = \log_2(600) + 4\log_2(2) = g + 4.$

Latvia: $\log_2(19200) = \log_2(600 \times 32) = \log_2(600) + \log_2(2^5) = \log_2(600) + 5\log_2(2) = g + 5.$

Denmark: $\log_2(38400) = \log_2(600 \times 64) = \log_2(600) + \log_2(2^6) = \log_2(600) + 6\log_2(2) = g + 6.$

(c) The new graph is



(d) The data are now much more linear. In general, you may want to take the logarithm of a variable when it's distribution is exponential – that is, when values tend to double or triple instead of increasing by additive amounts. Microeconomists, for example, almost always take the logarithm of income when they consider the income of randomly drawn people from the US population since the top income earners have many, many times the more income than people closer to the median.

7. (a)

$$= \ln(x^2 z) - \ln\left(\sqrt{e^y}\right)$$

$$= \ln(x^2) + \ln(z) - \ln\left((e^y)^{1/2}\right)$$

$$= 2\ln(x) + \ln(z) - \ln\left(e^{y/2}\right)$$

$$= 2\ln(x) + \ln(z) - \frac{y}{2}\ln(e)$$

$$= 2\ln(x) + \ln(z) - \frac{y}{2}.$$

(b) Writing out the terms of the long-products in the numerator and denominator:

$$\frac{2^1 \times 2^2 \times 2^3 \times \ldots \times 2^{99} \times 2^{100}}{2^2 \times 2^3 \times \ldots \times 2^{99} \times 2^{100}}.$$

Every factor is shared by the numerator and the denominator, except for $2^1 = 2$. Therefore every factor except 2 cancels, and the expression reduces to 2.

(c) Writing out the individual terms in the summation:

$$(5^1 - 5^0) + (5^2 - 5^1) + (5^3 - 5^2) + (5^4 - 5^3) + \ldots + (5^{N-1} - 5^{N-2}) + (5^N - 5^{N-1}).$$

If we drop the parentheses and change the order in which the terms are added and subtracted, we can rewrite the sum as

$$= -5^0 + 5^1 - 5^1 + 5^2 - 5^2 + 5^3 - 5^3 + \ldots + 5^{N-2} - 5^{N-2} + 5^{N-1} - 5^{N-1} + 5^N.$$

Every power of 5 is both added and subtracted, except for $-5^0 = -1$ and 5^N, which as the first and last terms to appear in the summation only appear once. Since every other term cancels out, we are left with

$$5^N - 1.$$

(d) First, observe that multiplication inside a logarithm can be rewritten as addition outside a logarithm. The same logic can be extended – long-products inside a logarithm can be rewritten as summations outside a logarithm:

$$\sum_{i=1}^{N} \ln(2e^{a_i}).$$

The log can be broken up into the sum of the log of each factor,

$$\sum_{i=1}^{N} \left(\ln(2) + \ln(e^{a_i}) \right),$$

the natural log cancels the exponential base of the second logarithm,

$$\sum_{i=1}^{N} \left(\ln(2) + a_i \right),$$

the summation can be distributed to each term,

$$\sum_{i=1}^{N} \ln(2) + \sum_{i=1}^{N} a_i,$$

and since the first logarithmic term does not contain the summation index i, this term is the same thing added to itself N times, so we can write it as

$$N \ln(2) + \sum_{i=1}^{N} a_i.$$

8. (a)

$$\begin{aligned} f(y) &= \log\left(p^y\right) + \log\left((1-p)^{1-y}\right) \\ &= y\log(p) + (1-y)\log(1-p). \end{aligned}$$

(b) A logarithm is only defined when the expression inside the parentheses is greater than zero. The expression

$$y\log(p) + (1-y)\log(1-p)$$

contains two logarithms, both of which contain p. The first logarithm is defined only when $p > 0$, and the second logarithm is defined only when $(1-p) > 0$, which implies that $p < 1$. These two conditions are both true only when $0 < p < 1$.

9. (a) Multiply both sides by $x+2$:

$$(3x-4)(x+2) = 2x^2 - 4x - 13.$$

Using FOIL on the left-hand side:

$$3x^2 + 2x - 8 = 2x^2 - 4x - 13.$$

Bringing all the terms to one side of the equation and combining like terms:

$$x^2 + 6x + 5 = 0.$$

Two numbers that add to 6 and multiply to 5 are 5 and 1, so the equation factors to

$$(x+5)(x+1) = 0.$$

The solutions are therefore

$$x = -5 \text{ and } x = -1.$$

Note that had $x = -2$ been a solution we would have had to disregard it as it would have placed a 0 in the denominator of the right-hand side of the original equation.

(b) To begin with, notice that we will have to disregard any solution that is less than $x = -13/4$, as this would place a negative value underneath the square root. First, square both sides of the equation:

$$4x + 13 = (x+2)^2,$$

$$4x + 13 = x^2 + 4x + 4.$$

Combining like terms:

$$0 = x^2 - 9.$$

Note that $x^2 - 9$ is a difference of squares, which factors to

$$0 = (x+3)(x-3),$$

so the solutions are

$$x = -3 \text{ and } x = 3.$$

$x = -3$ is a valid solution since it is not less than -13/4 = -3.25.

(c) First, use the rule of exponents that says to add the exponents of two factors that share the same base:

$$10^{3x^2+x} = 100.$$

Next, cancel the base of 10 by taking the common-log (the log with a base of 10) of both sides.

$$\log\left(10^{3x^2+x}\right) = \log(100),$$

$$3x^2 + x = \log(100).$$

Note that $100 = 10^2$, so the left-hand side reduces to

$$3x^2 + x = \log\left(10^2\right),$$

$$3x^2 + x = 2\log(10),$$
$$3x^2 + x = 2.$$

Bring all the terms to one side:

$$3x^2 + x - 2 = 0.$$

Check to see if the right-hand side factors neatly. In this quadratic expression, $A = 3$, $B = 1$, and $C = -2$. First calculate that $A \times C = -6$. The factor pairs of -6 are (1 and -6), (2 and -3), (3 and -2), and (6 and -1). The pair (3 and -2) adds to $B = 1$. So we break the middle term into

$$3x^2 + 3x - 2x - 2 = 0,$$

and group the expression into

$$(3x^2 + 3x) + (-2x - 2) = 0$$

$3x$ factors out of the first set of parentheses and -2 factors out of the second:

$$3x(x + 1) - 2(x + 1) = 0.$$

We can now factor $(x + 1)$ out of both terms:

$$(3x - 2)(x + 1) = 0.$$

So the solutions are $x = 2/3$ and $x = -1$.

(d) Multiply out the left-hand side:

$$x - x^2 < -6$$

In order to write the x^2 with a positive coefficient, multiply both sides by -1. Note that this action requires that we flip the inequality sign:

$$x^2 - x > 6$$

Subtract 6 from both sides:

$$x^2 - x - 6 > 0.$$

Two numbers that add to -1 and multiply to -6 are 2 and -3, so the left-hand side of the inequality factors to:

$$(x + 2)(x - 3) > 0.$$

The left-hand side is 0 when $x = -2$ or $x = 3$, but the question requires us to find the region in which the $(x + 2)(x - 3)$ is negative. We need to test values of x that are (1) less than -2, (2) between -2 and 3, and (3) greater than 3. First we consider $x = -3$:

$$(-3 + 2)(-3 - 3) = -1 \times -6 = 6,$$

so numbers in this region satisfy the inequality. Next we try $x = 0$:

$$(0 + 2)(0 - 3) = 2 \times -3 = -6,$$

so x values between -2 and 3 do not satisfy the inequality. Finally, we try $x = 4$:

$$(4+2)(4-3) = 6 \times 1 = 6,$$

so numbers in this region satisfy the inequality. Therefore the solution set consists of any x which is less than -2 or greater than 3.

10. (a) Here is one way to proceed. We start with

$$\ell(\mu) = \ln\left(\prod_{i=1}^{N} \frac{1}{\sqrt{2\pi}} e^{-.5(x_i-\mu)^2}\right).$$

Remember that $\log(ab) = \log(a) + \log(b)$, which means that multiplication inside a logarithm becomes addition outside a logarithm. That remains true even if we are multiplying a lot of terms together, and it still remains true if we represent this multiplication with a long product. So a long-product inside a logarithm becomes a summation outside the logarithm:

$$\ell(\mu) = \sum_{i=1}^{N} \ln\left(\frac{1}{\sqrt{2\pi}} e^{-.5(x_i-\mu)^2}\right).$$

Now we can deal with the terms inside the logarithm. First we break the logarithm up into

$$\ell(\mu) = \sum_{i=1}^{N} \left[\ln\left(\frac{1}{\sqrt{2\pi}}\right) + \ln\left(e^{-.5(x_i-\mu)^2}\right)\right].$$

One property of summations is that $\sum_i (a_i + b_i) = \sum_i a_i + \sum_i b_i$. That's just another way of saying that we can rearrange terms that are being added together in any order we like. So we can break up the summation as follows:

$$\ell(\mu) = \sum_{i=1}^{N} \ln\left(\frac{1}{\sqrt{2\pi}}\right) + \sum_{i=1}^{N} \ln\left(e^{-.5(x_i-\mu)^2}\right).$$

Let's work with the second term first. The natural-log cancels an exponential base of e:

$$\ell(\mu) = \sum_{i=1}^{N} \ln\left(\frac{1}{\sqrt{2\pi}}\right) + \sum_{i=1}^{N} \Big(-.5(x_i-\mu)^2\Big),$$

and by the same distribution property that says $(ab + ac) = a(b + c)$, the -.5 can be brought outside the sum

$$\ell(\mu) = \sum_{i=1}^{N} \ln\left(\frac{1}{\sqrt{2\pi}}\right) - .5\sum_{i=1}^{N} (x_i-\mu)^2.$$

Now let's work with the first term. We can rewrite the square root and the reciprocal as a power of $-1/2$:

$$\ell(\mu) = \sum_{i=1}^{N} \ln\Big((2\pi)^{-1/2}\Big) - .5\sum_{i=1}^{N} (x_i-\mu)^2.$$

The exponent can be brought outside the logarithm, and also outside the summation:

$$\ell(\mu) = -\frac{1}{2}\sum_{i=1}^{N} \ln(2\pi) - .5\sum_{i=1}^{N} (x_i-\mu)^2.$$

Notice that $\ln(2\pi)$ is a constant, and a constant that is added to itself N times is equal to N times the constant. So we can write the function as

$$\ell(\mu) = -\frac{1}{2} N \ln(2\pi) - .5 \sum_{i=1}^{N} (x_i - \mu)^2,$$

which can be rewritten as

$$\ell(\mu) = \frac{-\ln(2\pi)}{2} N - .5 \sum_{i=1}^{N} (x_i - \mu)^2.$$

(b) The logarithm turned the long-product into an expression that only contains a summation. The logarithm also canceled out the exponential base. In general, it is much easier to deal with addition than multiplication, and it is easier to deal with factors than exponents. Researchers prefer to take the log of their likelihood functions precisely because the logarithm turns multiplication into addition, and turns exponents into factors.