Chapter 2: Sets and Functions

1. (a) A function maps each input to one and only one output. The domain is the set of possible inputs, and the range is the set of possible outputs. Remember that a set is a collection of objects that are often, *but not necessarily*, numbers. In this case, the domain is the set of all humans, and the range is the set of all zombies. In the social sciences (in subfields other than the geopolitics of zombie apocalypses) we can use functions to map non-numbers to other non-numbers. For example, in game theoretical economics we map an actor's preferences to the actor's actions, in political science we can map a country's governmental regime type to its level of journalistic freedom, and in psychology we map certain compositions of neuro-chemicals to particular psychological disorders. Perhaps, for precision, we might measure these concepts with numbers, but there is nothing inherently numeric about these functions.

   (b) Originally, there is 1 zombie. After 1 day, there are 3. On day 2 there are $9 = 3 \times 3 = 3^2$. On day 3 there are $27 = 3 \times 3 \times 3 = 3^3$. In general, after x days, there are

$$f(x) = 3^x$$

   zombies.

   (c) We simply plug 28 into the function from part (b):

$$f(28) = 3^{28} = 22,876,790,000,000 \text{ [22.8 \textit{trillion}] zombies.}$$

   (d) We set $f(x) = 318,851,733$, and we solve for x:

$$3^x = 318,851,733.$$

   We take the logarithm, base-3, of both sides in order to cancel out the exponential base of 3:

$$\log_3(3^x) = \log_3(318,851,733),$$

$$x = \log_3(318,851,733).$$

   In order to get a computer or calculator to evaluate the value of this logarithm, we can convert it to a natural logarithm as follows:

$$x = \frac{\ln(318,851,733)}{\ln(3)} \approx 17.8 \text{ days.}$$

2. (a)

$$f(x) = \sqrt{\frac{x}{2}},$$

we plug in x^2 for x:

$$f(x^2) = \sqrt{\frac{(x^2)}{2}},$$

$$f(x^2) = \frac{\sqrt{x^2}}{\sqrt{2}},$$

$$f(x^2) = \frac{x}{\sqrt{2}}.$$

This answer is fine, but if you want to remove the square root term from the denominator, you can multiply the top and bottom of this fraction by $\sqrt{2}$:

$$f(x^2) = \frac{\sqrt{2}x}{2}.$$

(b)

$$f(x) = \sqrt{\frac{x}{2}}.$$

First we replace $f(x)$ with y:

$$y = \sqrt{\frac{x}{2}},$$

then we interchange x and y:

$$x = \sqrt{\frac{y}{2}},$$

and we solve for y:

$$x^2 = \frac{y}{2},$$

$$y = 2x^2.$$

Finally we substitute $f^{-1}(x)$ in for y:

$$f^{-1}(x) = 2x^2.$$

(c)

$$\begin{aligned}(f \circ g)(y) &= f(g(y)) = f(y^2 - 2y + 4)\\ &= \sqrt{\frac{y^2 - 2y + 4}{2}}.\end{aligned}$$

(d)

$$(g \circ f)(x) = g(f(x)) = g\left(\sqrt{\frac{x}{2}}\right)$$

$$= \left(\sqrt{\frac{x}{2}}\right)^2 - 2\left(\sqrt{\frac{x}{2}}\right) + 4$$
$$= \frac{x}{2} - 2\sqrt{\frac{x}{2}} + 4.$$

(e)

$$(f \circ f)(x) = f(f(x)) = f\left(\sqrt{\frac{x}{2}}\right)$$
$$= \sqrt{\frac{\sqrt{\frac{x}{2}}}{2}}$$
$$= \sqrt{\frac{\sqrt{\frac{x}{2}}}{2}}.$$

But square roots within square roots are pretty messy. So let's convert everything to exponents, remembering that exponents in the denominator can be expressed as negative exponents:

$$\left(\frac{\sqrt{\frac{x}{2}}}{2}\right)^{1/2}$$
$$= \left(\frac{(\frac{x}{2})^{1/2}}{2}\right)^{1/2}$$
$$= \left(\frac{x^{1/2}2^{-1/2}}{2}\right)^{1/2}$$
$$= \left(x^{1/2}2^{-1/2}\right)^{1/2} 2^{-1/2}$$
$$= x^{1/4}2^{-1/4}2^{-1/2}$$
$$= 2^{-3/4}x^{1/4}$$
$$= {}^4\sqrt{\frac{x}{2^3}}$$
$$= {}^4\sqrt{\frac{x}{8}}.$$

You didn't necessarily have to simplify this much, but I wanted to show how much the question could possibly be simplified.

(f)

$$(h \circ k)(w) = h(k(w)) = h(2\ln(w))$$
$$= e^{-2\ln(w)/2}$$
$$= e^{-\ln(w)}.$$

Recall that for any logarithm, we have the property that $-\ln(w) = \ln(1/w)$. So the equation becomes

$$e^{\ln(1/w)},$$

and now the exponential base and the natural logarithm cancel, leaving us with

$$\frac{1}{w}.$$

(g)

$$\begin{aligned}(k \circ h)(z) &= k(h(z)) = k(e^{-z/2}) \\ &= 2\ln(e^{-z/2}).\end{aligned}$$

The natural logarithm and the exponential base cancel each other out and we are left with

$$= 2(-z/2)$$

$$= -z.$$

3. (a) The function contains a logarithm, a square root, and a fraction. Only positive numbers can be placed inside a logarithm (you can't take the log of 0), and only non-negative numbers can be placed inside a square root (unlike a log, you can take the square-root of 0). Values of x that are strictly greater than 3 are allowed inside this particular logarithm, and values of x which are 5 or less are allowed inside this square root. Finally, we cannot have a fraction with 0 in the denominator, which occurs when $x = 5$, so 5 cannot be included in the domain. So all together, the function accepts real number values of x that are greater than 3 and less than 5. In set-builder notation, this statement is

$$\{x \in \mathbf{R} \mid 3 < x < 5\}.$$

(b) A fraction is 0 when the numerator is equal to 0. So we set

$$\ln(x-3) = 0.$$

We cancel out the natural logarithm by setting both sides as exponents with a base of e:

$$e^{\ln(x-3)} = e^0,$$

$$x - 3 = 1,$$

$$x = 4.$$

We have to check whether this value of x exists in the domain, and by our answer in part (a), it does. An alternative way to solve the problem would be to observe that $\ln(1) = 0$ and then to set $x - 3 = 1$.

(c) TRUE. I plug values of x which are closer and closer to 5 into the function:

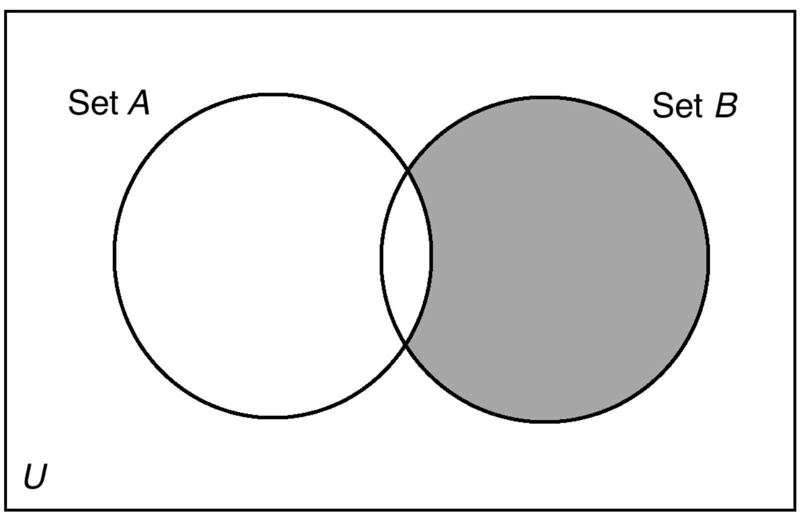
x	$f(x)$
4.9	2.03
4.99	6.88
4.999	21.90
4.9999	69.31
4.99999	219.19
4.999999	693.15
4.99999999999	219192.38

So it appears that the values of the function are definitely approaching ∞ as x gets closer to 5. On the other side, I hit the limit on the number of digits the spreadsheet program I'm using can handle before rounding, but the closest I can get to 3 is:

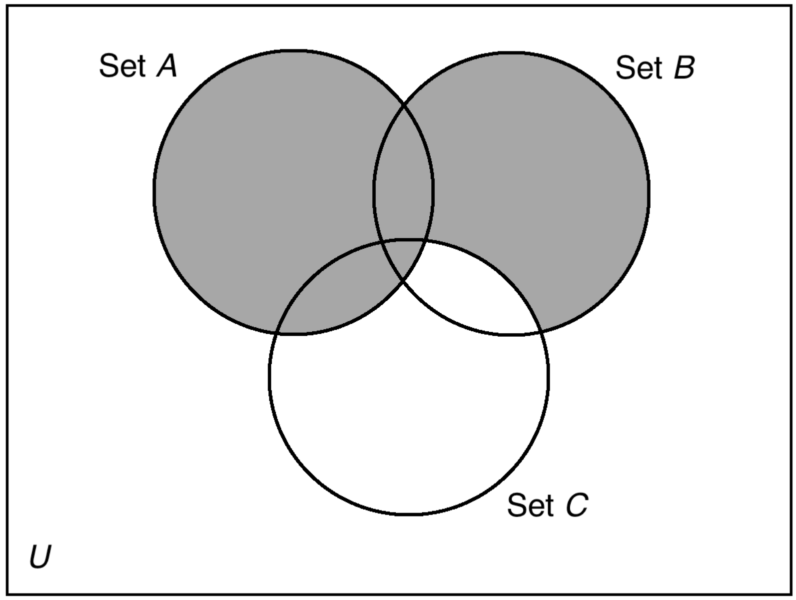
x	$f(x)$
3.1	-1.67
3.01	-3.26
3.001	-4.89
3.0001	-6.51
3.00001	-8.14
3.00000000000001	-22.78

While -22.78 is not indicative of a function whose values are exploding towards $-\infty$, there is also nothing in the behavior of the function to indicate that the decrease is leveling off as I approach 3. If I didn't hit the point where the spreadsheet program rounded by input to 3, I could have made this number as low as I liked.

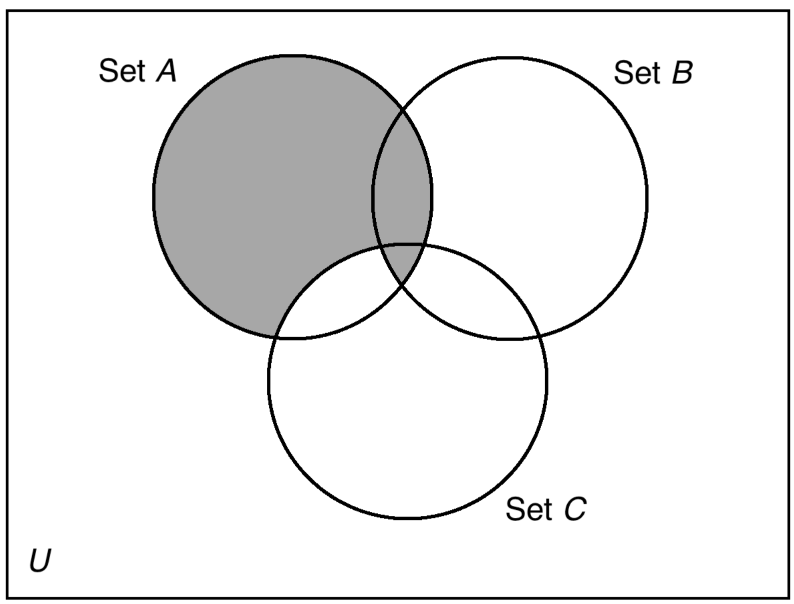
4. (a) As described in the text, $\tilde{A} \cap B$ is everything in set B that's also outside of set A:



(b) $A \cup (B \cap \tilde{C})$ is everything in set A, as well as everything that is both in B and outside of C:

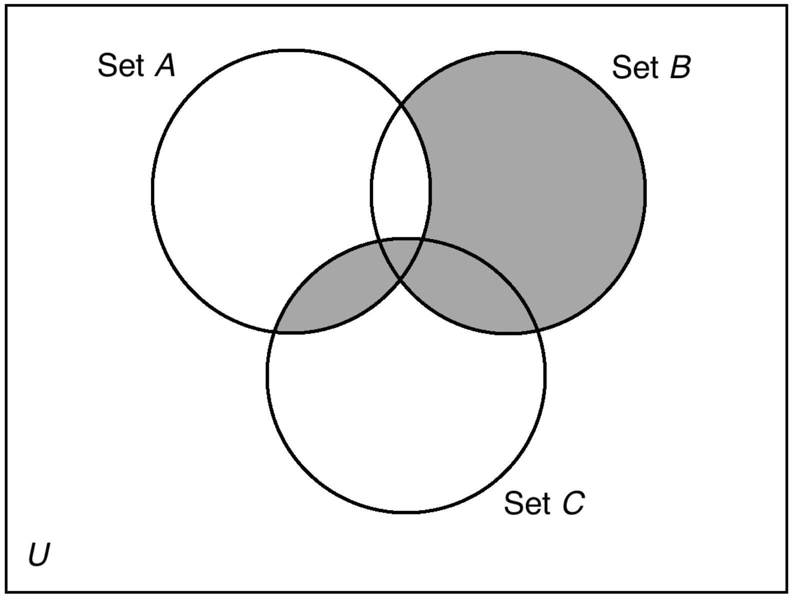


(c) $A \cap (B \cup \tilde{C})$ is everything that is both in set A and either in set B or outside C:



Here, the one part of set A that is not shaded is neither in set B, nor is outside C, so while it is inside A it does not satisfy the second requirement.

(d) $(\tilde{A} \cap B) \cup (A \cap C)$ consists of everything in B that's outside of A and everything in the intersection of A and C:



5. (a) All vegetables that aren't red (sorry tomatoes!)

   (b) All countries that are either non-democratic or economically underdeveloped, or both

   (c) All real numbers from 5 to 7, including 5, but not including 7

   (d) All real numbers strictly greater than 3

   (e) Integers that are at least as big as 2

   (f) A technical and literal interpretation is: all real numbers such that the number divided by 4 is an integer. A better, plain language is interpretation is: all real numbers that are divisible by 4.

6. (a) $\{x \in \mathbf{R} \mid -5 \leq x < 4\}$

   (b) $\{x \in \mathbf{R} \mid x > 12\}$

   (c) An integer is divisible by 3 if the fraction of that integer over 3 is also an integer (so no decimals): $\{x \in \mathbf{Z} \mid \frac{x}{3} \in \mathbf{Z}\}$

(d) Note, it's not actually necessary to solve this equation in order to express the set of solutions in set-builder notation. Simply put, this set is the set of real numbers that solve the equation $x^3 - 7x + 6 = 0$: $\{x \in \mathbf{R} \mid x^3 - 7x + 6 = 0\}$

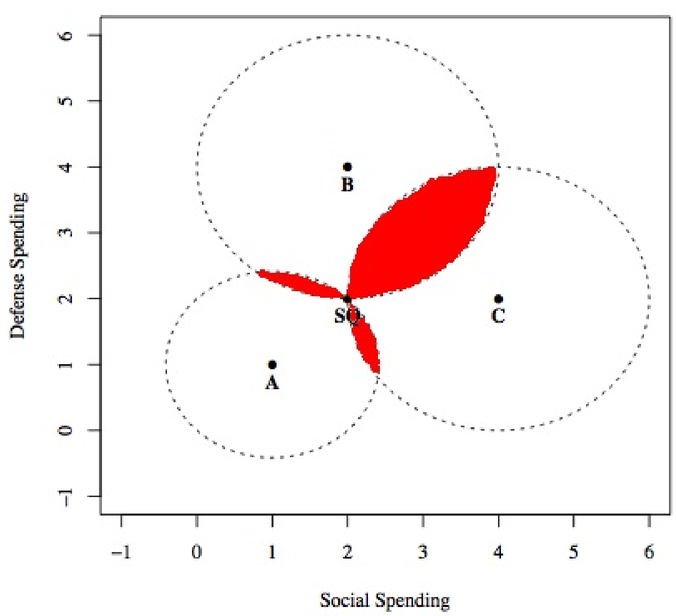
7. (a) A proposal will pass if it is voted for by:
A and B, or A and C, or B and C, or A and B and C.

We translate the above statement into set notation by using parentheses to separately denote the cases in which a bill will pass,
(A and B), or (A and C), or (B and C), or (A and B and C),

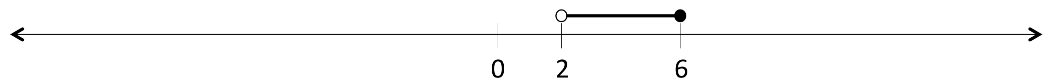
and then by replacing the "and" statements with intersections and the "or" statements with unions:
$(A \cap B) \cup (A \cap C) \cup (B \cap C) \cup (A \cap B \cap C)$.

Since the only bill that all three legislators can support $(A \cap B \cap C)$ is the same as the status quo, another acceptable answer is just
$(A \cap B) \cup (A \cap C) \cup (B \cap C)$.

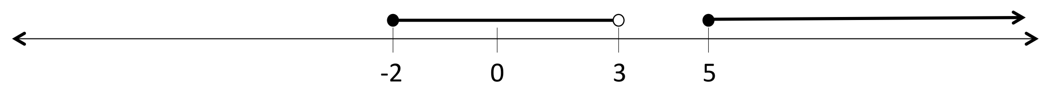
(b) The win-set is shaded in the graph below:



8. (a) This interval contains all real numbers between 2 and 6, not including 2 but including 6:



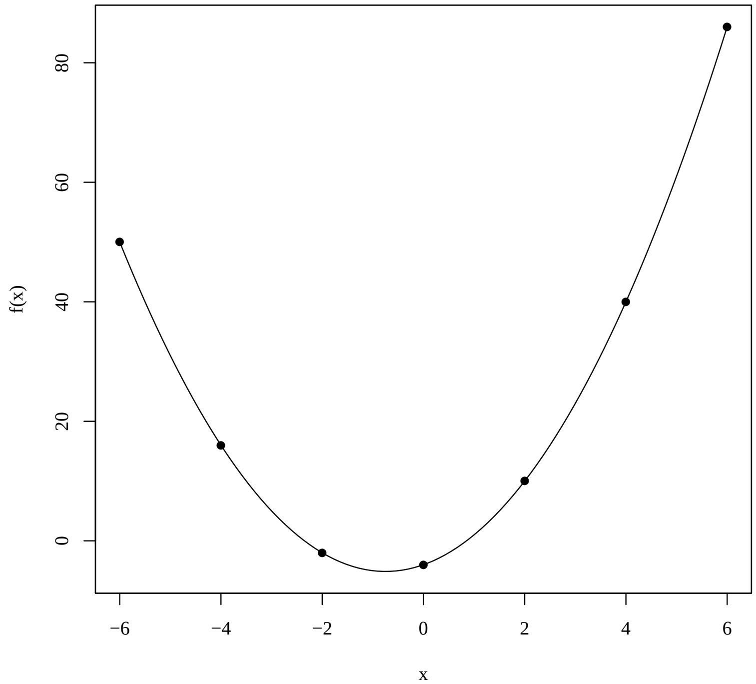
(b) This interval contains all real numbers between -2 and 3, including -2 but not including 3, as well as all real numbers greater than or equal to 5:



(c) For the following 4 graphs, it helps to list some points in a table by choosing a set of candidate x values and plugging them into the function. These points can be plotted, and these points suggest the overall shape of the graph. Some points that are on the graph of $f(x) = 2x^2 + 3x - 4$ are

x	$f(x)$
-6	50
-4	16
-2	-2
0	-4
2	10
4	40
6	86

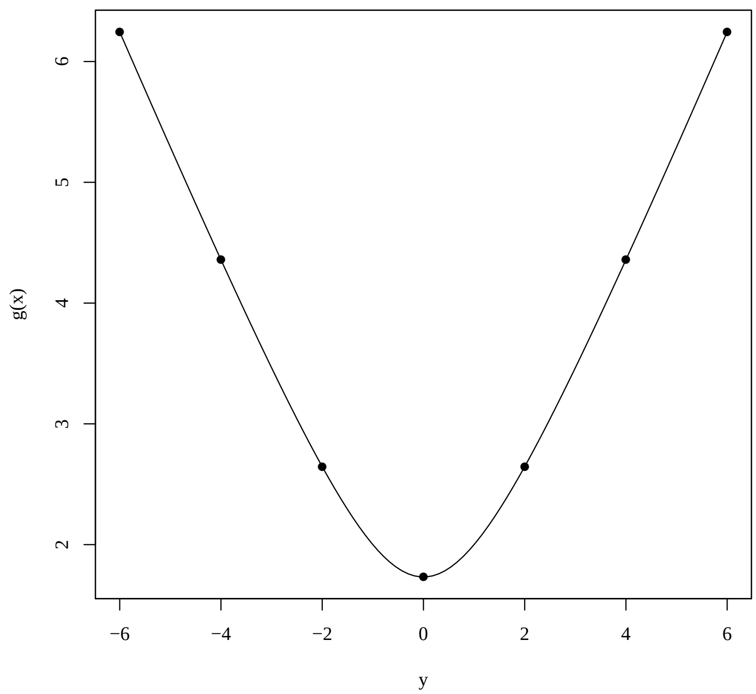
The graph of $f(x)$ is



(d) Some points that are on the graph of $g(y) = \sqrt{y^2 + 3}$ are

y	$g(y)$
-6	6.24
-4	4.36
-2	2.65
0	1.73
2	2.65
4	4.36
6	6.24

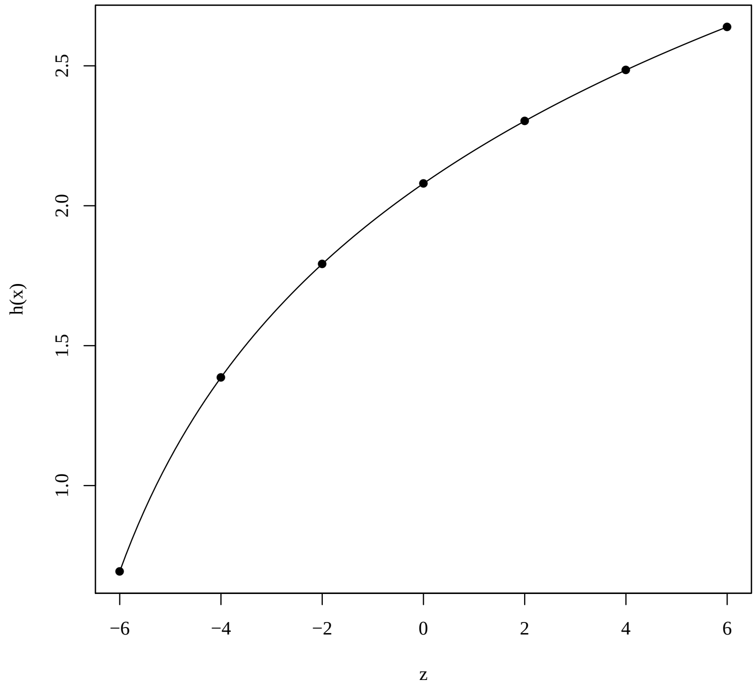
The graph of $g(y)$ is



(e) Some points that are on the graph of $h(z) = \ln(z + 8)$ are

z	$h(z)$
-6	0.69
-4	1.39
-2	1.79
0	2.08
2	2.30
4	2.48
6	2.64

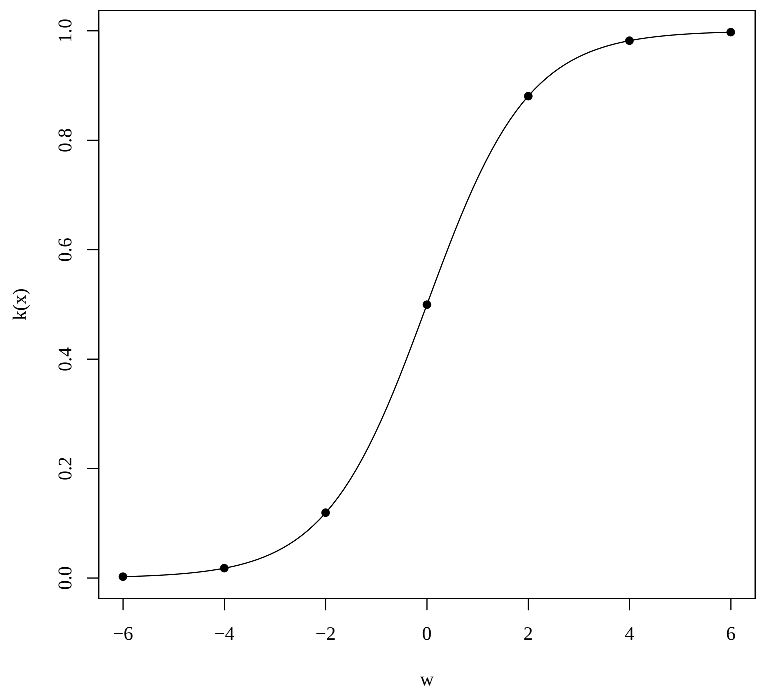
The graph of $h(z)$ is



(f) Some points that are on the graph of $k(w) = \frac{e^w}{1+e^w}$ are

w	$k(w)$
-6	0.00
-4	0.02
-2	0.12
0	0.50
2	0.88
4	0.98
6	1.00

The graph of $k(w)$ is



9. (a) The slope of a line that passes through the points (1,2) and (3,8) is

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{8-2}{3-1} = \frac{6}{2} = 3.$$

To find the y-intercept, we plug the slope and one of the points into

$$y = mx + b,$$

$$2 = (3)1 + b,$$

$$b = 2 - 3 = -1.$$

So this line is

$$y = 2x - 3.$$

(b) In this case, given a point and the y-intercept, we have enough information to plug into

$$y = mx + b$$

to solve for the slope. Plugging this information in gives us

$$1 = m(3) - 2,$$

$$3 = 3m,$$

$$m = 1.$$

So the line is

$$y = x - 2.$$

(c) In this case, given a point and the slope, we have enough information to plug into

$$y = mx + b$$

to solve for the y-intercept. Plugging this information in gives us

$$6 = -2(-3) + b,$$

$$6 = 6 + b,$$

$$b = 0.$$

So the line is

$$y = -2x.$$

(d) There are many examples of third degree polynomials with roots at -2, 1, and, 3. But the simplest example is

$$f(x) = (x + 2)(x - 1)(x - 3).$$

It's not necessary to multiply these terms out. Because three x terms are multiplied together, the largest power will be x^3. But to see this explicitly, begin by applying FOIL to the first two parenthetical terms,

$$f(x) = (x^2 + x - 2)(x - 3),$$

then distribute x and -3 to each of the quadratic terms,

$$f(x) = x^3 + x^2 - 2x - 3x^2 - 3x + 6,$$

and combine the like terms:

$$f(x) = x^3 - 2x^2 - 5x + 6.$$

(e) There are many examples of fifth degree polynomials with roots only at -4, 2, and, 5. But a simple example is

$$f(x) = (x + 4)^2(x - 2)^2(x - 5).$$

It's not necessary to multiply these terms out. Because two x^2 terms and one x term are multiplied together, the largest power will be x^5.

10. The symbols are individually translated as in the following table:

Symbol	Translation
U	The set of proposals that legislator 1 can make that will pass unanimously
$\equiv$	is equivalent to
$\{\ldots\}$	the set of
(x_2^1, x_3^1)	allocations to legislators 2 and 3 in the first proposal
$\mid$	such that
$x_j^1 \geq \delta v_j,\ j = 2, 3$	both legislator 2 and 3 receive allocations that are greater than or equal to the amount they expect from the next proposal, multiplied by the discount factor
$x_2^1 + x_3^1 \leq 1$	where the sum of legislator 2 and 3's allocations is less than or equal to 1
$\cap$	and
A	the allocations for legislators 1, 2, and 3 add up to exactly 1.

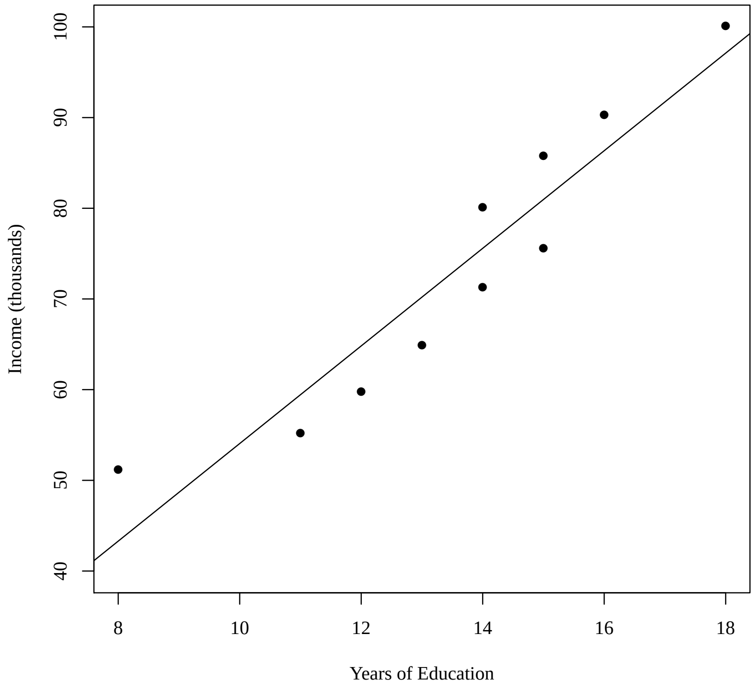
All together, the translation is

> "The set of proposals that legislator 1 can make that will pass unanimously is equivalent to the set of allocations to legislators 2 and 3 in the first proposal such that both legislator 2 and 3 receive allocations that are greater than or equal to the amount they expect from the next proposal multiplied by the discount factor, where the sum of legislator 2 and 3's allocations is less than or equal to 1, and the allocations for legislators 1, 2, and 3 add up to exactly 1."

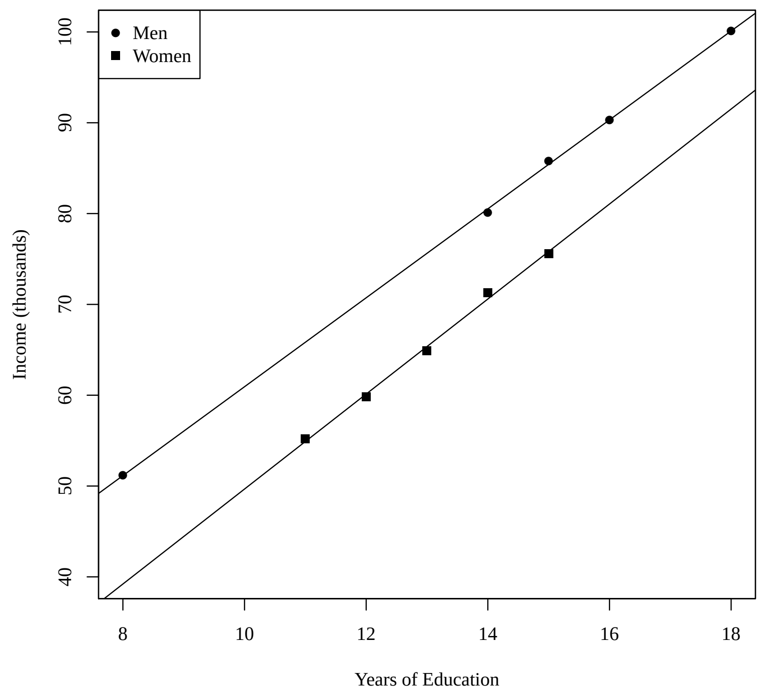
That's an awkward and wordy sentence. Like any English translation of a foreign language, direct word-for-word translations tend to be awkward and wordy. We can better express the spirit of the statement as something like

> "Any proposal by legislator 1 to allocate to legislators 2 and 3 at least as much as they would expect in later rounds, discounted for the penalty they incur by failing to pass a proposal right away, will pass the legislature unanimously provided that the allocations to legislators 2 and 3 do not exceed the amount of available resources and all available resources are allocated."

11. (a) The scatterplot of education and income, with the best fit line, is:



(b) Plotting the men with dots and the women with squares, and including separate best fit lines for men and for women:



The two lines are nearly parallel. After accounting for education, there appears to be a persistent wage gap between men and women that does not diminish with higher levels of education, although both men and women gain more income with more education.

(c) The constant is interpreted as follows: on average, a man (**Sex**=0) with no formal education (**Education**=0) makes \$11,356.

The coefficient on **Education** is interpreted as follows: an increase of one year in the number of years of formal education is associated with a \$4,939 increase in income, after accounting for the effect of sex on income.

The coefficient on **sex** is interpreted as follows: women, as compared to men, have incomes that are \$10,174 lower, on average, after accounting for the number of years of formal education.

(d) A women with 17 years of formal education has the following data: **Sex** = 1 and **Education**=17. Plugging these values into the regression model gives us:

$$\textbf{Income}_i = 11{,}356 + 4{,}939(17) - 10{,}174(1) + \varepsilon_i,$$

$$\textbf{Income}_i = 85{,}145 + \varepsilon_i.$$

The error ε_i represents the variation that we cannot explain – all of the ways we can be wrong about this prediction. But these errors are just as likely to be too high as too low, so on average, we expect a woman with 17 years of formal education to make \$85,145.