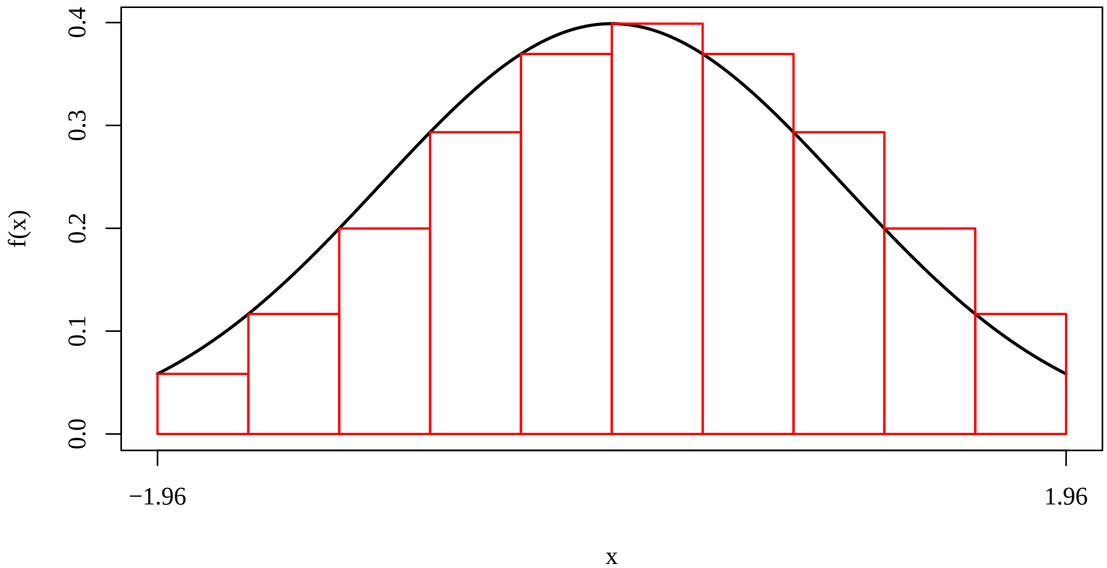
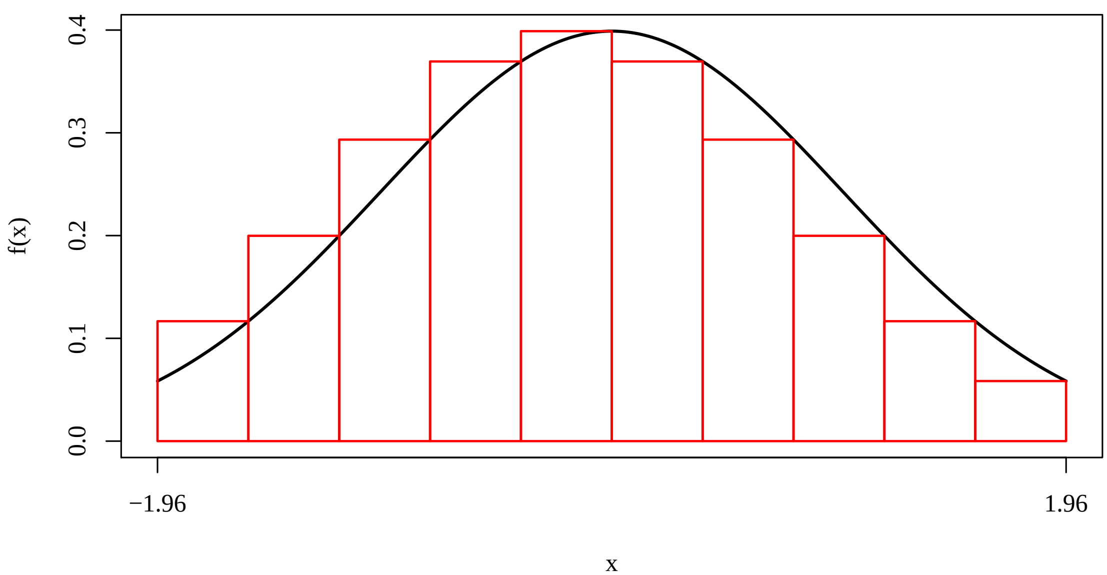
Chapter 6: Integration

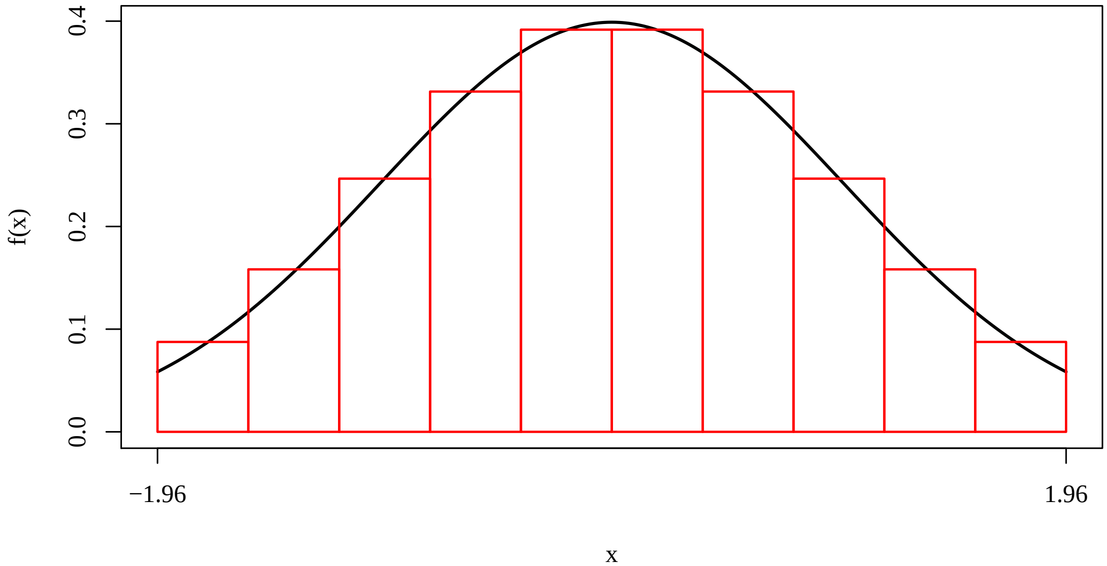
1. (a) The Riemann sums are graphed below. For the left Riemann sum, the top LEFT corner of each rectangle is the corner that touches the graph.



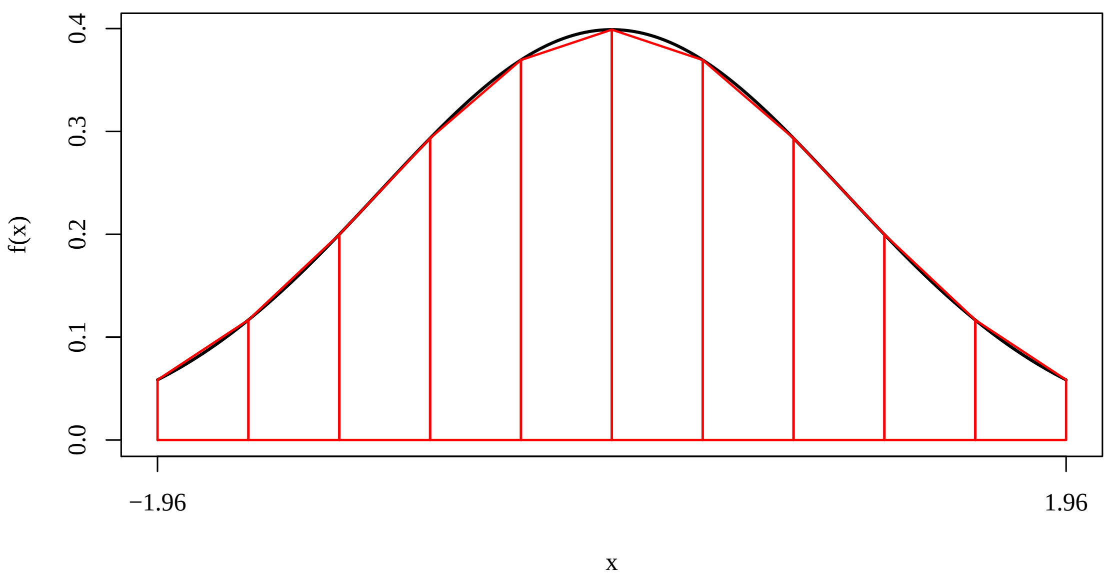
For the right Riemann sum, the top RIGHT corner of each rectangle is the corner that touches the graph.



For the midpoint Riemann sum, midpoint of the top of each rectangle touches the graph.



For the trapezoidal Riemann sum, both the top LEFT and top RIGHT corners of each trapezoid that touch the graph, and we draw a straight line between these two points.



(b) For the left Riemann sum we use the formula

$$A \approx \sum_{i=0}^{N-1} f\left(a + \frac{b-a}{N} i\right) \frac{b-a}{N},$$

and plug in $N = 10$ partitions, from $a = -1.96$ to $b = 1.96$, where f is

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-.5x^2}.$$

We get

$$\begin{aligned}
A &\approx \sum_{i=0}^{10-1} f\left(-1.96 + \frac{1.96-(-1.96)}{10} i\right) \frac{1.96-(-1.96)}{10} \\
&= \sum_{i=0}^{9} f(-1.96 + .392i).392 \\
&= .392 \sum_{i=0}^{9} f(-1.96 + .392i) \\
&= .392 \sum_{i=0}^{9} \frac{1}{\sqrt{2\pi}} e^{-.5(-1.96+.392i)^2} \\
&= \frac{.392}{\sqrt{2\pi}} \sum_{i=0}^{9} e^{-.5(-1.96+.392i)^2}.
\end{aligned}$$

For the right Riemann sum we use the formula

$$A \approx \sum_{i=1}^{N} f\left(a + \frac{b-a}{N} i\right) \frac{b-a}{N},$$

and plug in $N = 10$ partitions, from $a = -1.96$ to $b = 1.96$, where f is

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-.5x^2}.$$

We get

$$\begin{aligned}
A &\approx \sum_{i=1}^{10} f\left(-1.96 + \frac{1.96-(-1.96)}{10} i\right) \frac{1.96-(-1.96)}{10} \\
&= \sum_{i=1}^{10} f(-1.96 + .392i).392 \\
&= .392 \sum_{i=1}^{10} f(-1.96 + .392i) \\
&= .392 \sum_{i=1}^{10} \frac{1}{\sqrt{2\pi}} e^{-.5(-1.96+.392i)^2} \\
&= \frac{.392}{\sqrt{2\pi}} \sum_{i=1}^{10} e^{-.5(-1.96+.392i)^2}.
\end{aligned}$$

For the midpoint Riemann sum we use the formula

$$A \approx \sum_{i=1}^{N} f\left(a + \frac{b-a}{N}(i - .5)\right) \frac{b-a}{N},$$

and plug in $N = 10$ partitions, from $a = -1.96$ to $b = 1.96$, where f is

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-.5x^2}.$$

We get

$$\begin{aligned}
A &\approx \sum_{i=1}^{10} f\Bigg(-1.96 + \frac{1.96-(-1.96)}{10}(i-.5)\Bigg) \frac{1.96-(-1.96)}{10} \\
&= \sum_{i=1}^{10} f(-1.96 + .392(i-.5)).392 \\
&= .392 \sum_{i=1}^{10} f(-1.96 + .392(i-.5)) \\
&= .392 \sum_{i=1}^{10} f(-2.16 + .392i) \\
&= .392 \sum_{i=1}^{10} \frac{1}{\sqrt{2\pi}} e^{-.5(-2.16+.392i)^2} \\
&= \frac{.392}{\sqrt{2\pi}} \sum_{i=1}^{10} e^{-.5(-2.16+.392i)^2}.
\end{aligned}$$

And for the trapezoidal Riemann sum we use the formula

$$A \approx \sum_{i=0}^{N-1} \frac{\Big[f\Big(a + \frac{b-a}{N} i\Big) + f\Big(a + \frac{b-a}{N}(i+1)\Big)\Big]}{2} \frac{b-a}{N},$$

and plug in $N = 10$ partitions, from $a = -1.96$ to $b = 1.96$, where f is

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-.5x^2}.$$

We get

$$\begin{aligned}
A &\approx \sum_{i=0}^{9} \frac{\Big[f\Big(-1.96 + \frac{1.96-(-1.96)}{10} i\Big) + f\Big(-1.96 + \frac{1.96-(-1.96)}{10}(i+1)\Big)\Big]}{2} \frac{1.96-(-1.96)}{10} \\
&= \sum_{i=0}^{9} \frac{\Big[f\Big(-1.96 + .392i\Big) + f\Big(-1.96 + .392(i+1)\Big)\Big]}{2}.392 \\
&= \frac{.392}{2} \sum_{i=0}^{9} f\Big(-1.96 + .392i\Big) + f\Big(-1.96 + .392(i+1)\Big) \\
&= \frac{.196}{\sqrt{2\pi}} \sum_{i=0}^{9} e^{-.5(-1.96+.392i)^2} + e^{-.5(-1.96+.392(i+1))^2}.
\end{aligned}$$

(c) We simply write out the sums and evaluate them. For the left Riemann sum:

$$A \approx \frac{.392}{\sqrt{2\pi}} \sum_{i=0}^{9} e^{-.5(-1.96+.392i)^2}$$

$$= \frac{.392}{\sqrt{2\pi}} \Big[e^{-.5(-1.96+.392(0))^2} + e^{-.5(-1.96+.392(1))^2} + e^{-.5(-1.96+.392(2))^2} + e^{-.5(-1.96+.392(3))^2}$$
$$+ e^{-.5(-1.96+.392(4))^2} + e^{-.5(-1.96+.392(5))^2} + e^{-.5(-1.96+.392(6))^2} + e^{-.5(-1.96+.392(7))^2}$$
$$+ e^{-.5(-1.96+.392(8))^2} + e^{-.5(-1.96+.392(9))^2} \Big]$$
$$= 0.947.$$

For the right Riemann sum

$$A \approx \frac{.392}{\sqrt{2\pi}} \sum_{i=1}^{10} e^{-.5(-1.96+.392i)^2}$$

$$= \frac{.392}{\sqrt{2\pi}} \Big[e^{-.5(-1.96+.392(1))^2} + e^{-.5(-1.96+.392(2))^2} + e^{-.5(-1.96+.392(3))^2} + e^{-.5(-1.96+.392(4))^2}$$
$$+ e^{-.5(-1.96+.392(5))^2} + e^{-.5(-1.96+.392(6))^2} + e^{-.5(-1.96+.392(7))^2} + e^{-.5(-1.96+.392(8))^2}$$
$$+ e^{-.5(-1.96+.392(9))^2} + e^{-.5(-1.96+.392(10))^2} \Big]$$
$$= 0.947.$$

For the midpoint Riemann sum

$$A \approx \frac{.392}{\sqrt{2\pi}} \sum_{i=1}^{10} e^{-.5(-2.16+.392i)^2}$$

$$= \frac{.392}{\sqrt{2\pi}} \Big[e^{-.5(-2.16+.392(1))^2} + e^{-.5(-2.16+.392(2))^2} + e^{-.5(-2.16+.392(3))^2} + e^{-.5(-2.16+.392(4))^2}$$
$$+ e^{-.5(-2.16+.392(5))^2} + e^{-.5(-2.16+.392(6))^2} + e^{-.5(-2.16+.392(7))^2} + e^{-.5(-2.16+.392(8))^2}$$
$$+ e^{-.5(-2.16+.392(9))^2} + e^{-.5(-2.16+.392(10))^2} \Big]$$
$$= 0.951.$$

For the trapezoidal Riemann sum

$$A \approx \frac{.196}{\sqrt{2\pi}} \sum_{i=0}^{9} e^{-.5(-1.96+.392i)^2} + e^{-.5(-1.96+.392(i+1))^2}$$

$$= \frac{.196}{\sqrt{2\pi}} \Big[e^{-.5(-1.96+.392(0))^2} + 2e^{-.5(-1.96+.392(1))^2} + 2e^{-.5(-1.96+.392(2))^2} + 2e^{-.5(-1.96+.392(3))^2}$$
$$+ 2e^{-.5(-1.96+.392(4))^2} + 2e^{-.5(-1.96+.392(5))^2} + 2e^{-.5(-1.96+.392(6))^2} + 2e^{-.5(-1.96+.392(7))^2}$$
$$+ 2e^{-.5(-1.96+.392(8))^2} + 2e^{-.5(-1.96+.392(9))^2} + e^{-.5(-1.96+.392(10))^2} \Big]$$

Every added term, with the exception of the first and last, has a factor of 2. This factor represents that fact that the term appears twice. For example, the term $e^{-.5(-1.96+.392(5))^2}$ appears once when $i = 4$ and again when $i = 5$. The total of this sum is

$$= 0.947.$$

2. (a) There are no bounds on this integral, so this is an indefinite integral. Our purpose in calculating this integral is to find the anti-derivative of $f(x) = x^{100} + 3e^x - 7(4^x)$, that is, the function $F(x)$ whose derivative is $f(x)$.

First, we break up the integral over addition and subtraction:

$$F(x) = \int x^{100}\ dx + \int 3e^x\ dx - \int 7(4^x)\ dx.$$

Next we bring the constant factors outside the integrals:

$$F(x) = \int x^{100}\ dx + 3\int e^x\ dx - 7\int 4^x\ dx.$$

Finally we apply the rules for integrating power functions and exponential functions of base e and of base 4:

$$F(x) = \frac{x^{101}}{101} + 3e^x - 7\frac{4^x}{\ln(4)} + c,$$

where "+c" represents the constant that drops out of a derivative, and is added to any indefinite integral.

(b) This integral is definite because it has bounds, and our purpose in calculating this integral is to derive the area A between the curve of $f(x) = 5\sqrt{x} + \frac{3}{x^4}$ and the x-axis between $x = 1$ and $x = 9$.

The process for solving a definite integral is nearly the same as the process for solving an indefinite integral. As before, we start by breaking the integral up over addition,

$$A = \int_1^9 5\sqrt{x}\ dx + \int_1^9 \frac{3}{x^4}\ dx,$$

and we bring constant factors outside the integrals:

$$A = 5\int_1^9 \sqrt{x}\ dx + 3\int_1^9 \frac{1}{x^4}\ dx.$$

To simplify each integral, we can rewrite the square root as an exponent of 1/2, and we can rewrite the reciprocated exponent as a negative exponent:

$$A = 5\int_1^9 x^{1/2}\ dx + 3\int_1^9 x^{-4}\ dx.$$

Now we can apply the power rule to each integral and write the bounds to the right of the anti-derivative:

$$\begin{aligned} A &= 5\left(\frac{2}{3}x^{3/2}\right) + 3\left(\frac{-1}{3}x^{-3}\right)\Bigg|_1^9 \\ &= \frac{10}{3}x^{3/2} - \frac{1}{x^3}\Bigg|_1^9. \end{aligned}$$

Finally we plug in the upper-bound, plug in the lower-bound, and subtract. When we work with definite integrals, we do not write "+c" in the anti-derivative because it will cancel out when we subtract the anti-derivative at each bound:

$$\begin{aligned} A &= \left(\frac{10}{3}9^{3/2} - \frac{1}{9^3}\right) - \left(\frac{10}{3}1^{3/2} - \frac{1}{1^5}\right) \\ &= \left(\frac{10}{3}27 - \frac{3}{729}\right) - \left(\frac{10}{3} - 1\right) \\ &= \left(90 - \frac{3}{729}\right) - \frac{7}{3} = 87\frac{2}{3}. \end{aligned}$$

(c) This is a definite integral, but because it has an infinite bound, it is also an improper integral. We begin by using a limit to rewrite the integral with bounds that we can work with:

$$\lim_{K\to\infty} \int_2^K \frac{12}{x^2}\ dx.$$

We will solve the definite integral treating K as if it were finite. Then we will solve the limit once we've solved the integral. We can rewrite the integrand with a negative exponent,

$$\lim_{K\to\infty} \int_2^K 12x^{-2}\ dx,$$

bring the 12 outside the integral (and the limit too),

$$12 \lim_{K\to\infty} \int_2^K x^{-2}\ dx,$$

and apply the power rule of integration,

$$\begin{aligned} &12 \lim_{K\to\infty} \frac{x^{-1}}{-1}\Big|_2^K \\ &= 12 \lim_{K\to\infty} \frac{-1}{x}\Big|_2^K \\ &= 12 \lim_{K\to\infty} \frac{-1}{K} + \frac{1}{2}. \end{aligned}$$

Finally, note that as K goes to infinity, the fraction $\frac{-1}{K}$ approaches zero. So in the limit we can replace this fraction with 0, leaving us with

$$12\left(\frac{1}{2}\right) = 6.$$

(d) This problem involves the derivative of a definite integral. Since integrals and derivatives are inverse operations, they cancel out – but since the derivative is taken with respect to a variable other than x that exists in the bounds, we apply the first fundamental theorem of calculus for bounds:

$$\frac{d}{dy} \int_a^{g(y)} f(x)\ dx = f\Big(g(y)\Big)g'(y).$$

We don't have to find any anti-derivatives. We simply plug the upper bound y^2 into the integrand, and satisfy the chain rule by multiplying the function by the derivative of y^2. The solution is

$$\begin{aligned} &f\Big(g(y)\Big)g'(y) \\ &= \sqrt{y^2}(2y) = 2y^2. \end{aligned}$$

(e) First we multiply the integral by -1 so that the bounds are reversed:

$$-\frac{d}{dz} \int_{10}^{\sqrt{z}+\ln(z)} e^x\ dx.$$

Now we apply the first fundamental theorem of calculus for bounds:

$$-\frac{d}{dz}\int_a^{g(z)} f(x)\ dx = -f\Big(g(z)\Big)g'(z).$$

We don't have to calculate an anti-derivative, we only need to calculate $f\Big(g(z)\Big)$ and $g'(z)$:

$$\begin{aligned}&-f\Big(g(z)\Big)g'(z)\\ &= -e^{\sqrt{z}+\ln(z)}\left(\frac{1}{2\sqrt{z}}+\frac{1}{z}\right).\end{aligned}$$

3. (a) There are two polynomials in the integrand. The second polynomial, the one taken to the 7th power, is a degree-10 polynomial. When a degree-10 polynomial is differentiated, the result is a degree-9 polynomial. Since the first polynomial in the integrand is degree-9, it makes sense to try to set the second polynomial as u to see if du removes the first polynomial. So,

$$u = 5x^{10} - 25x^4 + 15x,$$

which means that the derivative is

$$\begin{aligned}\frac{du}{dx} &= 50x^9 - 100x^3 + 15,\\ \frac{1}{5}du &= (10x^9 - 20x^3 + 3)dx,\end{aligned}$$

which does in fact contain the first polynomial. Next we substitute u and $\frac{1}{5}du$ into the integral:

$$\begin{aligned}&\int (10x^9 - 20x^3 + 3)(5x^{10} - 25x^4 + 15x)^7\ dx\\ &= \int (5x^{10} - 25x^4 + 15x)^7\Big[(10x^9 - 20x^3 + 3)\ dx\Big]\\ &= \int u^7\left[\frac{1}{5}du\right]\\ &\frac{1}{5}\int u^7\ du.\end{aligned}$$

This integral can be solved with the power rule of integration:

$$\begin{aligned}&\frac{1}{5}\frac{u^8}{8} + c\\ &\frac{u^8}{40} + c.\end{aligned}$$

Finally, we substitute back for u:

$$\frac{(5x^{10} - 25x^4 + 15x)^8}{40} + c.$$

(b) The trick behind u-substitution is to find part of the integrand whose derivative resembles some factor inside the integrand. In this case, if we set

$$u = x^2 - 2,$$

then

$$\frac{du}{dx} = 2x.$$

Next we multiply both sides by dx, and divide both sides by 2 (to bring all constant factors with du instead of dx):

$$\frac{1}{2}\ du = x\ dx.$$

For a definite integral, we have two options for handling the bounds. First, we can plug each bound into our equation for u, and derive new bounds. If we pursue this option, we won't need to substitute back in for u at the end of the problem. The new lower bound is

$$u(0) = (0)^2 - 2 = -2,$$

and the new upper bound is

$$u(3) = (3)^2 - 2 = 9 - 2 = 7.$$

Now we can substitute u, $\frac{1}{2}du$, and the new bounds into the integral:

$$\int_0^3 xe^{x^2-2}\ dx = \frac{1}{2}\int_{-2}^{7} e^u\ du$$

$$= \frac{1}{2}e^u\Big|_{-2}^{7} = \frac{e^7 - e^{-2}}{2} = 548.25.$$

The second option for dealing with the bounds is to find the anti-derivative of the integral with u, substitute back in for u, and calculate the area under the curve using the *old* bounds. This anti-derivative (ignoring "+c" because this is a definite integral) is

$$e^u.$$

Substituting back in for u and applying the bounds gives us:

$$\frac{1}{2}e^{x^2-2}\Big|_0^3 = \frac{e^{3^2-2} - e^{0^2-2}}{2} = \frac{e^7 - e^{-2}}{2} = 548.25.$$

(c) If we set

$$u = 4x^2 + 6x - 11,$$

then

$$\frac{du}{dx} = 8x + 6, \quad du = 8x + 6\ dx, \quad \frac{1}{2}du = 4x + 3\ dx.$$

So by choosing this particular u, the numerator drops out of the function when we make the substitution. That's exactly the purpose of u-substitution. Substituting for $4x^2 + 6x - 11$ and for $(4x + 3)dx$, the integral now becomes

$$\frac{1}{2}\int \frac{1}{u}\ du,$$

$$= \frac{1}{2}\ln(|u|) + c.$$

Substituting back for u, this becomes

$$\frac{1}{2}\ln(|4x^2 + 6x - 11|) + c.$$

(d) It makes sense to first try setting the expression inside the square root to u,

$$u = x^2 + 1,$$

so that the derivative is

$$\frac{du}{dx} = 2x,$$
$$du = 2x\ dx.$$

The new lower bound is

$$u(0) = (0)^2 + 1 = 1,$$

and the new upper bound is

$$u(2) = (2)^2 + 1 = 5.$$

Substituting u, du, and the new bounds into the integral gives us

$$\begin{aligned}
&\int_1^5 \sqrt{u}\ du \\
&= \int_1^5 u^{\frac{1}{2}}\ du \\
&= \left.\frac{u^{\frac{3}{2}}}{\frac{3}{2}}\right|_1^5 \\
&= \left.\frac{2}{3}u^{\frac{3}{2}}\right|_1^5 \\
&= \frac{2}{3}\left(5^{\frac{3}{2}} - 1^{\frac{3}{2}}\right) \\
&= \frac{2}{3}\left(\sqrt{125} - 1\right) = 6.79.
\end{aligned}$$

4. (a) The integrand is the product of two functions: x and $\sqrt{x+5}$. Of these two functions, x definitely becomes simpler when differentiated, so we denote

$$u = x.$$

We will integrate the other function, so we denote it as

$$dv = \sqrt{x+5}\ dx.$$

Note, it is not necessary to see right away *how* we will integrate the other function at this point. We simply need to choose u and dv and to give integration by parts a chance. It is often the case that one selection of u and dv won't work, so we try a new way to write these functions. Even very experienced mathematicians have to try integration by parts again and again, so don't get frustrated.

The derivative of $u = x$ is

$$\frac{du}{dx} = 1.$$

Solving for du:

$$du = dx.$$

To find v, we take the indefinite integral of the right-hand side of the equation for dv:

$$v = \int \sqrt{x+5}\ dx.$$

To solve this integral, let's apply u-substitution by setting

$$w = x + 5,$$

$$\frac{dw}{dx} = 1, \qquad dw = dx.$$

Substituting w and dw into the integral gives us

$$v = \int \sqrt{w}\ dw$$

$$= \int w^{1/2}\ dw$$

$$= \frac{2}{3}w^{3/2}.$$

Finally we substitute back for w, and we avoid writing "+c" since this problem will eventually involve bounds. Therefore v is

$$v = \frac{2}{3}(x+5)^{3/2}.$$

Plugging these elements as well as the bounds $a = 1$ and $b = 4$ into the above formula gives us

$$\int_1^4 x\sqrt{x+5}\ dx$$

$$= x\left(\frac{2}{3}(x+5)^{3/2}\right)\Bigg|_1^4 - \int_1^4 \frac{2}{3}(x+5)^{3/2}\ dx.$$

We can simplify this equation slightly without solving it:

$$\left(\frac{2}{3}x(x+5)^{3/2}\right)\Bigg|_1^4 - \frac{2}{3}\int_1^4 (x+5)^{3/2}\ dx.$$

We have two steps left to take. First we have to solve the remaining integral, then we have to plug the bounds into the total anti-derivative and calculate the area under the curve. Consider the integral:

$$\int (x+5)^{3/2}\ dx.$$

Since this integral is part of a larger definite integral, we hold off on dealing with the bounds until the very end of the problem. We can solve this integral through u-substitution (using w instead of u to avoid confusion with the u we used earlier in this problem). Set

$$w = (x+5), \quad \text{so that } \tfrac{dw}{dx} = 1, \quad \text{and } dw = dx.$$

Substituting w and dw into the integral gives us:

$$\int w^{3/2}\ dw$$

$$\frac{2}{5}w^{5/2}.$$

Substituting back in for w gives us

$$\frac{2}{5}(x+5)^{5/2},$$

and plugging this anti-derivative into the integration by parts equation gives us

$$\int_1^4 x\sqrt{x+5}\ dx$$

$$= \left(\frac{2}{3}x(x+5)^{3/2}\right) - \frac{2}{3}\left(\frac{2}{5}(x+5)^{5/2}\right)\Big|_1^4$$

$$= \frac{2}{3}x(x+5)^{3/2} - \frac{4}{15}(x+5)^{5/2}\Big|_1^4$$

$$= \left(\frac{2}{3}(4)(4+5)^{3/2} - \frac{4}{15}(4+5)^{5/2}\right) - \left(\frac{2}{3}(1)(1+5)^{3/2} - \frac{4}{15}(1+5)^{5/2}\right)$$

$$= \left(\frac{2}{3}(4)(9)^{3/2} - \frac{4}{15}(9)^{5/2}\right) - \left(\frac{2}{3}(1)(6)^{3/2} - \frac{4}{15}(6)^{5/2}\right)$$

$$= \left(\frac{2}{3}(4)(27) - \frac{4}{15}(243)\right) - \left(\frac{2}{3}(6)^{3/2} - \frac{4}{15}(6)^{5/2}\right)$$

$$= 20.92.$$

(b) Let

$$u = 3x, \qquad dv = e^x\ dx,$$

which means that

$$du = 3\ dx, \qquad dv = \int e^x\ dx = e^x.$$

Applying the integration by parts formula, we get

$$\int u\ dv = uv - \int v\ du$$

$$= 3xe^x - \int e^x(3\ dx)$$

$$= 3xe^x - 3\int e^x\ dx$$

$$= 3xe^x - 3e^x + c.$$

(c) When using integration by parts, we have to identify two functions multiplied together, identify the factor that simplifies when we take its derivative and set it to u, and set the other factor to dv. In this case, the integrand is the product of two functions, x and $\ln(x)$. The choice of which function to set to u is tricky. Our first instinct should be to set $u = x$, but then we would have to integrate $dv = \ln(x)\ dx$, which is tricky. Instead, if we set $u = \ln(x)$ then we can easily integrate $dv = x\ dx$. If we make this choice, then

$$du = \frac{1}{x}dx,$$

and

$$v = \int x\ dx = \frac{x^2}{2}.$$

We hold off evaluating the area under the curve between the bounds until the end of the problem. Now that we have u, du, v, and dv, we plug these functions into the integration by parts formula:

$$\int u\ dv = uv - \int v\ du$$

$$
\begin{aligned}
&= \ln(x)\frac{x^2}{2} - \int \frac{x^2}{2}\frac{1}{x}\ dx \\
&= \frac{x^2\ln(x)}{2} - \int \frac{x}{2}\ dx \\
&= \frac{x^2\ln(x)}{2} - \frac{x^2}{4}.
\end{aligned}
$$

We do not write "$+c$" because we are working with a definite integral. Now that we have evaluated the anti-derivative, we plug in the bounds:

$$
\begin{aligned}
&\left.\frac{x^2\ln(x)}{2} - \frac{x^2}{4}\right|_1^e \\
&= \frac{e^2\ln(e)}{2} - \frac{e^2}{4} - \left(\frac{1^2\ln(1)}{2} - \frac{1^2}{4}\right) \\
&= \frac{e^2}{2} - \frac{e^2}{4} + \frac{1}{4} \\
&= \frac{e^2+1}{4}.
\end{aligned}
$$

(d) Note that x^2 becomes simpler if we take the derivative, and e^x is easily integrated. So let's set

$$u = x^2, \qquad dv = e^x\ dx,$$

which implies

$$du = 2x\ dx, \qquad v = \int e^x\ dx = e^x,$$

where we hold off on writing "$+c$" until the end of the problem. Plugging these quantities into the integration by parts formula, we get

$$
\begin{aligned}
\int u\ dv &= uv - \int v\ du \\
&= x^2e^x - \int e^x(2x\ dx) \\
&= x^2e^x - 2\int xe^x\ dx.
\end{aligned}
$$

Now we have to contend with the integral $\int xe^x\ dx$, which requires doing integration by parts a second time. This time, let

$$u = x, \qquad dv = e^x\ dx,$$

so that

$$du = \ dx, \qquad v = \int e^x\ dx = e^x.$$

Then $\int xe^x\ dx$ becomes

$$
\begin{aligned}
\int u\ dv &= uv - \int v\ du \\
&= xe^x - \int e^x\ dx
\end{aligned}
$$

$$= xe^x - e^x.$$

Plugging this integral in for $\int xe^x \ dx$ above, we get a solution for the entire indefinite integral:

$$\int x^2 e^x \ dx = x^2 e^x - 2(xe^x - e^x) + c$$
$$= x^2 e^x - 2xe^x + 2e^x + c.$$

5. (a) This integral does not require u-substitution or integration by parts because it can be evaluated directly from basic rules of integration. First we can break the integral up over addition:

$$\int_{\ln(8)}^{3\sqrt{167}} x^2 + e^x \ dx = \int_{\ln(8)}^{3\sqrt{167}} x^2 \ dx + \int_{\ln(8)}^{3\sqrt{167}} e^x \ dx$$
$$= \frac{x^3}{3}\bigg|_{\ln(8)}^{3\sqrt{167}} + e^x\bigg|_{\ln(8)}^{3\sqrt{167}}$$
$$= \left(\frac{(3\sqrt{167})^3}{3} - \frac{\ln(8)^3}{3}\right) + \left(e^{3\sqrt{167}} - e^{\ln(8)}\right)$$
$$= \left(\frac{167}{3} - \frac{\ln(8)^3}{3}\right) + \left(e^{3\sqrt{167}} - 8\right) = 291.05.$$

(b) Observe that the derivative of $\ln(x)$ is $\frac{1}{x}$, which is a factor of the integrand. So let's apply u-substitution where

$$u = \ln(x), \quad \tfrac{du}{dx} = \tfrac{1}{x}, \quad du = \frac{1}{x}dx.$$

We can also recalculate the bounds using this formula for u. The new lower bound is

$$u(e^2) = \ln(e^2) = 2,$$

and the new upper bound is

$$u(e^4) = \ln(e^4) = 4.$$

Substituting for $\ln(x)$, $\frac{1}{x}dx$, and the upper and lower bounds, the definite integral becomes

$$\int_2^4 u \ du$$
$$= \frac{u^2}{2}\bigg|_2^4$$
$$= \frac{4^2}{2} - \frac{2^2}{2} = 8 - 2 = 6.$$

(c) The problem looks a lot like part (b), but note that using u-substitution in which $u = \ln(x)$ won't cancel out the denominator as it did before. It is necessary to solve this problem through integration by parts instead. How were you supposed to know that? You probably had to go through a frustrating process in which you tried various u-substitutions before giving up on that particular tool and trying versions of integration by parts until something worked. That is NOT a problem. In fact, that is what I had to do to solve this problem too. The thing about integration by parts and u-substitution is that they are tools, and integrals are problems in which the correct tools are not always obvious. There is no single process for solving many integrals, and they can be frustrating to even the most experienced mathematicians. If you struggled to find the correct method, you have joined a long and noble tradition.

Anyway, noting that we can differentiate $\ln(x)$ and that we can integrate $\frac{1}{x^5} = x^{-5}$, we set

$$u = \ln(x), \qquad dv = x^{-5}\ dx,$$

which means that

$$du = \frac{1}{x}\ dx, \qquad v = \int x^{-5}\ dx = \frac{x^{-4}}{-4} = \frac{-1}{4x^4}.$$

Plugging these functions into the formula for integration by parts, we get

$$\begin{aligned}\int u\ dv &= uv - \int v\ du \\ &= \ln(x)\frac{-1}{4x^4} - \int \frac{-1}{4x^4}(\frac{1}{x}\ dx) \\ &= \frac{-\ln(x)}{4x^4} - \int \frac{-1}{4x^5}\ dx \\ &= \frac{-\ln(x)}{4x^4} + \frac{1}{4}\int x^{-5}\ dx \\ &= \frac{-\ln(x)}{4x^4} + \frac{1}{4}\frac{x^{-4}}{-4} + c \\ &= \frac{-\ln(x)}{4x^4} - \frac{1}{16x^4} + c.\end{aligned}$$

(d) First we can break the integral up over addition:

$$\int e^x\ dx + \int \sqrt{x}\ dx$$

Then we can solve each indefinite integral. The first integral evaluates to

$$\int e^x\ dx = e^x + c.$$

The second integral can be rewritten as

$$\int x^{1/2}\ dx,$$

so we apply the power rule for integrals:

$$\int x^{1/2}\ dx = \frac{x^{3/2}}{3/2} + c = \frac{2}{3}x^{3/2} + c.$$

Combining both integrals, we get

$$\int e^x\ dx + \int \sqrt{x}\ dx = e^x + \frac{2}{3}x^{3/2} + c.$$

(e) First we find the anti-derivative as we would while solving an indefinite integral. The integral breaks up over addition and subtraction:

$$\int_0^{10} y^2\ dy - \int_0^{10} 10y\ dy + \int_0^{10} 25\ dy.$$

Leaving the bounds aside for now, the first integral evaluates to

$$\int y^2\ dy = \frac{y^3}{3}.$$

We leave off "+c" since the constant term ultimately would cancel out once we plug in the bounds. The constant factor 10 can be brought outside the second integral

$$10 \int y\ dy = 10\frac{y^2}{2} = 5y^2.$$

The third integral evaluates to

$$\int 25\ dy = 25y.$$

So the entire anti-derivative is

$$\frac{y^3}{3} - 5y^2 + 25y.$$

The area underneath the curve from 0 to 10 is

$$\frac{y^3}{3} - 5y^2 + 25y\bigg|_0^{10}$$

$$= \left(\frac{(10)^3}{3} - 5(10)^2 + 25(10)\right) - \left(\frac{(0)^3}{3} - 5(0)^2 + 25(0)\right)$$

$$= \left(\frac{1000}{3} - 500 + 250\right) - 0 = 83\frac{1}{3}.$$

(f) We can observe that the integrand contains a part whose derivative (without its coefficient) exists elsewhere in the integrand. Specifically, the function underneath the square root, $x^3 + 2$, which becomes a multiple of x^2 when we take its derivative. That indicates that we should use u-substitution. First set u to the term under the square root:

$$u = x^3 + 2.$$

When we take the derivative, we get,

$$\frac{du}{dx} = 3x^2,$$

$$du = 3x^2\ dx,$$

$$\frac{1}{3}du = x^2\ dx.$$

We can also change the bounds:

$$u(2) = 2^3 + 2 = 10, \quad \text{and} \quad u(4) = 4^3 + 2 = 66.$$

Next, we substitute u for $x"3 + 2$, we substitute $1/3du$ for x^2dx, we substitute 10 for the lower bound, and we substitute 66 for the upper bound. The problem becomes

$$\frac{1}{3}\int_{10}^{66} \sqrt{u}\ du$$

$$\begin{aligned}
&= \frac{1}{3}\int_{10}^{66} u^{1/2}\ du \\
&= \frac{1}{3}\frac{2}{3}u^{3/2}\Big|_{10}^{66} \\
&= \frac{2}{9}(66^{3/2} - 10^{3/2}) = 112.13.
\end{aligned}$$

(g) This integrand is the product of two functions, so we can consider using integration by parts. We set one function as u and take its derivative to find du, and we set the other function as dv and take its anti-derivative to find v, then we plug all of these functions into

$$\int u\ dv = uv - \int v\ du,$$

and with any luck, the remaining integral is easier to solve. Let's set

$$u = \ln(x) \quad \text{and} \quad dv = x^2\ dx,$$

then du and v are

$$du = \frac{1}{x}\ dx \quad \text{and} \quad v = \int x^2\ dx = \frac{x^3}{3}.$$

Plugging into the equations by parts formula, this becomes

$$\begin{aligned}
&\frac{x^3\ln(x)}{3} - \frac{1}{3}\int \frac{x^3}{x}\ dx \\
&\frac{x^3\ln(x)}{3} - \frac{1}{3}\int x^2\ dx \\
&\frac{x^3\ln(x)}{3} - \frac{1}{3}\frac{x^3}{3} + c \\
&x^3\left(\frac{\ln(x)}{3} - \frac{1}{9}\right) + c.
\end{aligned}$$

(h) Since the integral has an infinite upper bound, this is an improper integral. We can rewrite the integral as

$$\lim_{A\to\infty}\int_1^A \frac{3x^2+2x+1}{(x^3+x^2+x+1)^2}\ dx$$

and solve as if A were finite. We will take the limit only after solving the integral. Note that the derivative of the term inside the square in the denominator is exactly equal to the numerator. So let's use u-substitution by setting

$$\begin{aligned}
u &= x^3+x^2+x+1, \\
du &= 3x^2+2x+1\ dx.
\end{aligned}$$

Let's not substitute the bounds, instead we will substitute back in for u once we've taken the integral. The integral (without the bounds) now becomes

$$\int \frac{1}{u^2}\ du = \int u^{-2}\ du = \frac{u^{-1}}{-1} = \frac{-1}{u}.$$

Substituting back for u and considering the bounds, this becomes

$$\left. \frac{-1}{x^3 + x^2 + x + 1} \right|_1^A$$

$$\frac{-1}{A^3 + A^2 + A + 1} - \frac{-1}{1^3 + 1^2 + 1 + 1}$$

$$\frac{-1}{A^3 + A^2 + A + 1} + \frac{1}{4}.$$

Finally, we take the limit as $A \to \infty$:

$$\lim_{A \to \infty} \frac{-1}{A^3 + A^2 + A + 1} + \frac{1}{4}.$$

Note that as A gets larger and larger, so does the polynomial $A^3 + A^2 + A + 1$. The denominator therefore approaches $-\infty$, and the fraction $1/-\infty$ approaches zero. Replacing the limit with 0, the integral is equal to $1/4$.

6. (a) The domain of the function contains values of x between 0 and 1. First we have to demonstrate that the function is never negative in this domain. When $x = 0$, the function is

$$f(0) = \frac{3\sqrt{0}}{2} = 0,$$

which isn't positive, but also isn't negative. Every other x in the domain is positive, implying that the function is positive as well. Therefore $f(x)$ is never negative for any x in the domain.

Second, we have to demonstrate that the function integrates to 1 over its domain. That means we have to demonstrate that the following definite integral is equal to 1:

$$\int_0^1 \frac{3\sqrt{x}}{2}\, dx$$

$$= \frac{3}{2} \int_0^1 \sqrt{x}\, dx$$

$$= \frac{3}{2} \int_0^1 x^{1/2}\, dx$$

$$= \left. \frac{3}{2} \left(\frac{2}{3} x^{3/2} \right) \right|_0^1$$

$$= \left. x^{3/2} \right|_0^1 = (1)^{3/2} - 0^{3/2} = 1.$$

Therefore $f(x)$ is a PDF.

(b) We already did most of the necessary work in part (a). We simply plug in new bounds.

To calculate the probability that the student gets an A, we find the area under the curve of the PDF from .9 to 1:

$$P(A) = P(.9 < x < 1) = x^{3/2}\Big|_{.9}^{1} = 1^{3/2} - .9^{3/2} = 0.146.$$

To calculate the probability that the student gets a B, we find the area under the curve of the PDF from .8 to .9:

$$P(B) = P(.8 < x < .9) = x^{3/2}\Big|_{.8}^{.9} = .9^{3/2} - .8^{3/2} = 0.138.$$

To calculate the probability that the student gets a C, we find the area under the curve of the PDF from .7 to .8:

$$P(C) = P(.7 < x < .8) = x^{3/2}\Big|_{.7}^{.8} = .8^{3/2} - .7^{3/2} = 0.130.$$

To calculate the probability that the student gets a D, we find the area under the curve of the PDF from .6 to .7:

$$P(D) = P(.6 < x < .7) = x^{3/2}\Big|_{.6}^{.7} = .7^{3/2} - .6^{3/2} = 0.121.$$

Finally, to calculate the probability that the student gets an F, we find the area under the curve of the PDF from 0 to .6:

$$P(F) = P(x < .6) = x^{3/2}\Big|_{0}^{.6} = .6^{3/2} - 0^{3/2} = 0.465.$$

(c) This problem involves two challenges: first we have to plug the relevant information into the formula for an expected value,

$$E(x) = \int_{-\infty}^{\infty} xf(x)\ dx,$$

and second we have to solve that definite integral. The function is $f(x) = \frac{3\sqrt{x}}{2}$, and the domain is $x \in [0, 1]$. Plugging this information into the formula gives us

$$\begin{aligned}
E(x) = \int_{-\infty}^{\infty} xf(x)\ dx &= \int_{0}^{1} x\frac{3\sqrt{x}}{2}\ dx \\
&= \frac{3}{2}\int_{0}^{1} x\sqrt{x}\ dx \\
&= \frac{3}{2}\int_{0}^{1} x(x^{1/2})\ dx \\
&= \frac{3}{2}\int_{0}^{1} x^{3/2}\ dx \\
&= \frac{3}{2}\left(\frac{2}{5}x^{5/2}\right)\Big|_{0}^{1} \\
= \frac{3}{5}\left(x^{5/2}\right)\Big|_{0}^{1} &= \frac{3}{5}\left(1^{5/2} - 0^{5/2}\right) = \frac{3}{5} = 0.6.
\end{aligned}$$

So the average score is 60%. Ouch.

(d) The second part of the variance formula is the square of .6:

$$V(x) = E(x^2) - E(x)^2,$$

$$V(x) = E(x^2) - .36.$$

So the challenge is finding $E(x^2)$, which involves plugging into the given formula and solving the resulting definite integral. The formula is

$$E(x^2) = \int_{-\infty}^{\infty} x^2 f(x)\,dx = \int_0^1 x^2 \frac{3\sqrt{x}}{2}\,dx$$

$$= \frac{3}{2}\int_0^1 x^2\sqrt{x}\,dx$$

$$= \frac{3}{2}\int_0^1 x^2(x^{1/2})\,dx$$

$$= \frac{3}{2}\int_0^1 x^{5/2}\,dx$$

$$= \frac{3}{2}\left(\frac{2}{7}x^{7/2}\right)\Big|_0^1$$

$$= \frac{3}{7}\left(x^{7/2}\right)\Big|_0^1 = \frac{3}{7}\left(x^{7/2}\right)\Big|_0^1 = \frac{3}{7}\left(1^{7/2} - 0^{7/2}\right) = \frac{3}{7} = 0.429.$$

Plugging this value back into the formula for variance gives us

$$V(x) = E(x^2) - E(x)^2 = .429 - .36 = 0.069.$$

Finally, the standard deviation is the square root of the variance:

$$SD(x) = \sqrt{V(x)} = \sqrt{0.069} = 0.262.$$

Therefore the mean score is 60%, and the standard deviation is 26.2%.

7. A function is a PDF if it is non-negative everywhere in its domain and if it integrates to 1 over its domain. First, note that $f(x) = \frac{3}{7x^4}$ is positive for all x between $\frac{1}{2}$ and 1, so the first condition is met. Next, we evaluate

$$\int_{1/2}^1 \frac{3}{7x^4}\,dx$$

$$= \frac{3}{7}\int_{1/2}^1 \frac{1}{x^4}\,dx$$

$$= \frac{3}{7}\int_{1/2}^1 x^{-4}\,dx$$

$$= \frac{3}{7}\frac{x^{-3}}{-3}\Big|_{1/2}^1$$

$$= \frac{-1}{7x^3}\Big|_{1/2}^1$$

$$= \frac{-1}{7(1)^3} - \frac{-1}{7(1/2)^3}$$

$$
\begin{aligned}
&= \frac{-1}{7} + \frac{1}{7(1/8)} \\
&= \frac{-1}{7} + \frac{1}{7/8} \\
&= \frac{-1}{7} + \frac{8}{7} = \frac{7}{7} = 1.
\end{aligned}
$$

8. Note that e^{x+1} is positive for all x in the domain $x \in [0, k]$ no matter what k is. Our task here is to choose the value of k that makes the integral evaluate to 1. We solve the definite integral as we always would, and plug k into the anti-derivative in the last step. Then we set this expression equal to 1 and solve for k. The definite integral is:
$$\int_0^k e^{x+1}\, dx.$$
We can solve this integral using u-substitution. Let $u = x + 1$. Then $du = dx$, the lower bound is $u(0) = 0 + 1 = 1$, and the upper bound is $u(k) = k + 1$. Substituting, we get
$$
\begin{aligned}
&\int_1^{k+1} e^u\, du \\
&= e^u \Big|_1^{k+1} \\
&= e^{k+1} - e^1 \\
&= e^{k+1} - e.
\end{aligned}
$$
Now we set this expression equal to 1 and solve for k:
$$
\begin{aligned}
e^{k+1} - e &= 1, \\
e^{k+1} &= e + 1, \\
k + 1 &= \ln(e + 1), \\
k &= \ln(e + 1) - 1.
\end{aligned}
$$
So $g(x)$ is a PDF on the domain $x \in [0, \ln(e + 1) - 1]$.

9. (a) If $f(x)$ is a PDF, then the formula for a CDF is
$$F(x) = \int_{-\infty}^x f(x)\, dx,$$
where the lower bound of $-\infty$ stands in for whatever the lower bound of the distribution happens to be. In this case, the formula for the CDF is
$$
\begin{aligned}
F(x) &= \int_0^x \lambda e^{-\lambda x}\, dx \\
&= \lambda \int_0^x e^{-\lambda x}\, dx.
\end{aligned}
$$

We can solve this integral using u-substitution, where $u = -\lambda x$, $du = -\lambda\ dx$, and therefore $\frac{-1}{\lambda} du = dx$. The lower bound becomes

$$u(0) = -\lambda(0) = 0,$$

and the upper bound becomes

$$u(x) = -\lambda x.$$

Substituting, the integral becomes

$$\begin{aligned}
&\lambda\left(\frac{-1}{\lambda}\right)\int_0^{-\lambda x} e^u\ du \\
&= -e^u\Big|_0^{-\lambda x} \\
&= -e^{-\lambda x} - (-e^0) \\
&= -e^{-\lambda x} + 1 \\
&= 1 - e^{-\lambda x},
\end{aligned}$$

which is the correct formula for the exponential distribution CDF.

(b) The CDF of a function at a value c tells you the probability that a randomly drawn value from that distribution is less than or equal to c,

$$F(c) = P(x \leq c),$$

and the probability that a randomly drawn value is between two values c and d is given by

$$P(d \leq x \leq c) = P(x \leq c) - P(x \leq d) = F(c) - F(d).$$

So in this case, the probability that x is between 30 and 50 is

$$\begin{aligned}
&F(50) - F(30) \\
&= 1 - e^{-\lambda(50)} - (1 - e^{-\lambda(30)}).
\end{aligned}$$

In the case of Claudio Cioffi-Revilla's model, $\lambda = -0.023$, so the probability becomes

$$= 1 - e^{-0.023(50)} - (1 - e^{-0.023(30)}) = 0.185.$$

(c) If $f(x)$ is a PDF, then the formula for the expected value of this PDF is

$$E(x) = \int_{-\infty}^{\infty} x\ f(x)\ dx$$

where the bounds $-\infty$ and ∞ stand in for the bounds of the domain of the PDF. Since the lower bound of the domain is 0, we replace $-\infty$ with 0. Let's replace the upper bound ∞ with a value k for now. At the end of the problem we will plug a very large value in for k to see if the expression simplifies. The integral now becomes

$$\begin{aligned}
E(x) &= \int_0^k x(.023e^{-.023x})\ dx \\
&= \int_0^k .023xe^{-.023x}\ dx.
\end{aligned}$$

We have to evaluate this integral using integration by parts, and let's not deal with the bounds until the end of the problem. Let

$$u = .023x, \qquad dv = e^{-.023x}\ dx,$$

which means that

$$du = .023\ dx.$$

In order to find v, we have to solve

$$v = \int e^{-.023x}\ dx,$$

which requires u-substitution. Unfortunately, the naming conventions muddle the fact that we have a u from integration by parts and a u from u-substitution. So let's call this method w-substitution in this case. Let

$$w = -.023x, \quad dw = -.023\ dx, \quad \frac{-1}{.023}\ dw = dx.$$

Substituting, this integral becomes

$$v = \frac{-1}{.023}\int e^{w}\ dw = \frac{-1}{.023}e^{w}.$$

Substituting back in for w, we get

$$v = \frac{-1}{.023}e^{-.023x}.$$

Now that we have u, v, du, and dv, we plug these functions into the formula for integration by parts:

$$\int u\ dv = uv - \int v\ du$$

$$= .023x\left(\frac{-1}{.023}e^{-.023x}\right) - \int \frac{-1}{.023}e^{-.023x}(.023\ dx)$$

$$= -xe^{-.023x} + \int e^{-.023x}\ dx.$$

The remaining integral is the same one we solved above using w-substitution, so the whole expression becomes

$$-xe^{-.023x} - \frac{1}{.023}e^{-.023x}.$$

Since this integral is definite with bounds at 0 and k, we plug in the bounds:

$$-xe^{-.023x} - \frac{1}{.023}e^{-.023x}\Big|_0^k$$

$$= \left(-ke^{-.023k} - \frac{1}{.023}e^{-.023k}\right) - \left(-0e^{-.023(0)} - \frac{1}{.023}e^{-.023(0)}\right)$$

$$= \left(-ke^{-.023k} - \frac{1}{.023}e^{-.023k}\right) + \frac{1}{.023}.$$

Now, since k is standing in for ∞, we can get an idea about how this expression evaluates by thinking about plugging very large numbers in for k. When k is very large, the quantity $e^{-.023k}$ is infinitesimally close to 0, which implies that both terms inside the parentheses are 0. Therefore the whole expression evaluates to

$$E(x) = \frac{1}{.023} = 43.5 \text{ weeks}.$$

(d) The variance of a PDF is given by

$$V(x) = E(x^2) - E(x)^2.$$

In part (c) we found that $E(x) = 43.5$, so the variance becomes

$$\begin{aligned} V(x) &= E(x^2) - (43.5)^2 \\ &= E(x^2) - 1890.4. \end{aligned}$$

So we just have to find $E(x^2)$, which is given by the following formula:

$$E(x^2) = \int_{-\infty}^{\infty} x^2 \ f(x) \ dx.$$

As in part (c), the integral specifically can be written as

$$\begin{aligned} E(x^2) &= \int_0^k x^2 \ (.023e^{-.023x}) \ dx \\ &= \int_0^k .023x^2 e^{-.023x} \ dx. \end{aligned}$$

To solve this integral, we use integration by parts where

$$u = .023x^2, \qquad dv = e^{-.023x} \ dx,$$

which implies that

$$du = .046x \ dx,$$

and, from the work we did in part (c),

$$v = \int e^{-.023x} \ dx = \frac{-1}{.023} e^{-.023x}.$$

Plugging u, v, du, and dv into the integration by parts formula we get

$$\begin{aligned} \int u \ dv &= uv - \int v \ du \\ &= .023x^2 \left(\frac{-1}{.023} e^{-.023x} \right) - \int \frac{-1}{.023} e^{-.023x} (.046x \ dx) \\ &= -x^2 e^{-.023x} - \int \frac{-.046}{.023} x e^{-.023x} \ dx. \\ &= -x^2 e^{-.023x} \Big|_0^k + 2 \int_0^k x e^{-.023x} \ dx. \end{aligned}$$

Remember that in part (c) we found that

$$E(x) = \int_0^{\infty} .023 x e^{-.023x} \ dx = 43.5.$$

That means that we can divide both sides by .023 to find a solution to $\int x e^{-.023x} \ dx$:

$$\int_0^{\infty} x e^{-.023x} \ dx = \frac{43.5}{.023} = 1890.4.$$

Substituting this number into $E(x^2)$ gives us

$$E(x^2) = \lim_{k \to \infty} -x^2 e^{-.023x} \Big|_0^k + 2(1890.4).$$

Note that a very large value of k will make the quantity $e^{-.023k}$ approach 0. Therefore the entire first term approaches 0, and

$$E(x^2) = 2(1890.4) = 3780.7.$$

Finally, we plug $E(x^2)$ into the formula for the variance of the distribution:

$$V(x) = E(x^2) - 1890.4$$

$$V(x) = 3780.7 - 1890.4 = 1890.4.$$

The standard deviation is the square root:

$$SD(x) = \sqrt{1890.4} = 43.5 \text{ weeks.}$$

The fact that the mean equals the standard deviation is no accident. That is a property of the exponential distribution. More recent work employs models that do not make such a restrictive assumption about the mean and standard deviation of the dependent variable.

10. (a) The expected value of income is

$$\text{E(Income)} = \text{E}\Big(\alpha{+}\beta_1\text{Gender} + \beta_2\text{Education} + \beta_3(\text{Gender}{\times}\text{Education}) + \varepsilon\Big).$$

We can break this expected value up over addition,

$$\text{E}\Big(\alpha{+}\beta_1\text{Gender} + \beta_2\text{Education} + \beta_3(\text{Gender}{\times}\text{Education})\Big) + \text{E}(\varepsilon).$$

The problem tells us to treat the coefficients and variables as if they are constant. That means that the whole expression $\alpha{+}\beta_1\text{Gender} + \beta_2\text{Education} + \beta_3(\text{Gender}{\times}\text{Education})$ is a constant, and therefore that the first expected value is the expected value of a constant. Since the expected value of a constant is just the constant itself, the whole expression becomes

$$\alpha{+}\beta_1\text{Gender} + \beta_2\text{Education} + \beta_3(\text{Gender}{\times}\text{Education}) + \text{E}(\varepsilon).$$

For the second expected value, we know that a regression error term is a random variable with a mean of zero. Since expected value is another name for the mean, we can replace this term with zero. The expected value of income is

$$\text{E(Income)} = \alpha{+}\beta_1\text{Gender} + \beta_2\text{Education} + \beta_3(\text{Gender}{\times}\text{Education}).$$

(b) The derivative of E(Income) with respect to gender is

$$\frac{dE(\text{Income})}{d\text{Gender}} = \frac{d}{d\text{Gender}}\Big(\alpha{+}\beta_1\text{Gender} + \beta_2\text{Education} + \beta_3(\text{Gender}{\times}\text{Education})\Big).$$

It might be difficult to connect this equation to the derivatives we took in chapter 4. If might help to replace every occurrence of the word gender with x, every occurrence of the word income with y, and every occurrence of the word education with another symbol like z. Then the equation is

$$\frac{dE(y)}{dx} = \frac{d}{dx}(\alpha + \beta_1 x + \beta_2 z + \beta_3 xz).$$

The derivative is

$$\frac{dE(y)}{dx} = \beta_1 + \beta_3 z,$$

and plugging words back in for the symbols gives us

$$\frac{dE(\text{Income})}{d\text{Gender}} = \beta_1 + \beta_3\text{Education}.$$

(c) If $\alpha = 10000$, $\beta_1 = 15000$, $\beta_2 = 5000$, and $\beta_3 = -500$ then the derivative of E(Income) with respect to gender is

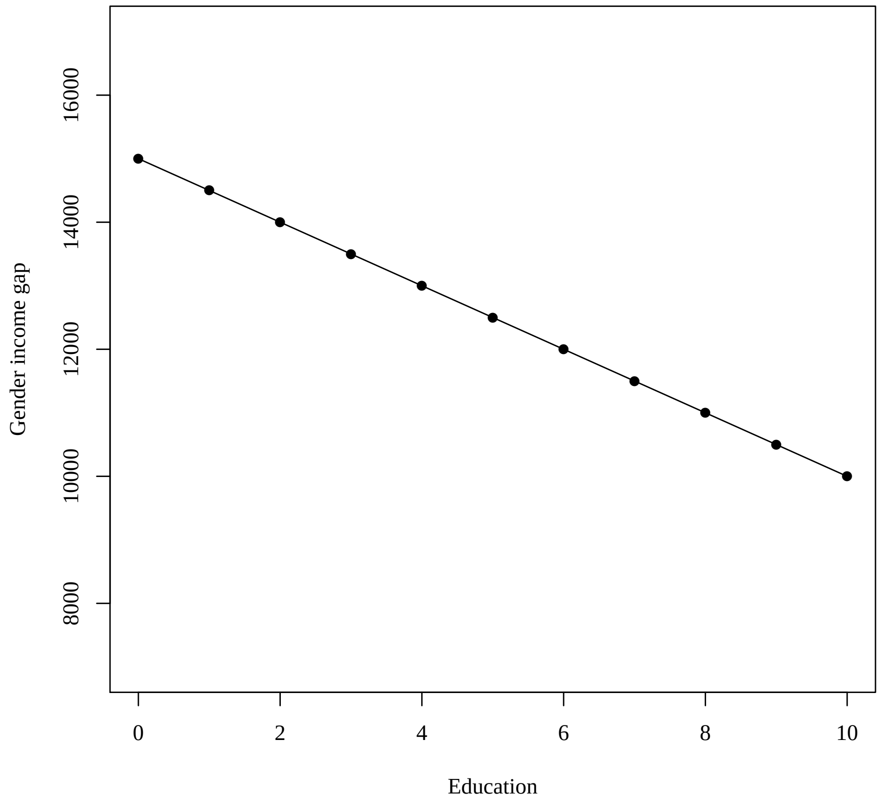
$$\frac{dE(\text{Income})}{d\text{Gender}} = 15000 \text{ - } 500\text{Education}.$$

The derivative of E(Income) with respect to gender is one way to measure the gender income gap. We simply plug in the level of education and compute $\frac{dE(\text{Income})}{d\text{Gender}}$ using the above equation. These values are listed in the table below:

At education level	The gender income gap is
0	15000-500(0)=15000
1	15000-500(1)=14500
2	15000-500(2)=14000
3	15000-500(3)=13500
4	15000-500(4)=13000
5	15000-500(5)=12500
6	15000-500(6)=12000
7	15000-500(7)=11500
8	15000-500(8)=11000
9	15000-500(9)=10500
10	15000-500(10)=10000

The table tells us that while the gender income gap is decreasing with higher levels of education, the gap exists everywhere. Even among people with the highest education level men make $10,000 more than women, on average.

(d) The graph of these values is below:



(e) The variance of the gender gap in expected income is

$$V\left(\frac{dE(\text{Income})}{d\text{Gender}}\right) = V(\beta_1 + \beta_3 \text{Education}).$$

Here we are treating β_1 and β_3 as the random variables and education as constant. The rule regarding the variance of sums of random variables is

$$V(aX + bY) = a^2V(x) + b^2V(y) + 2abCov(x, y).$$

Substituting 1 for a, β_1 for x, education for b, and β_3 for y, the variance is

$$V\left(\frac{dE(\text{Income})}{d\text{Gender}}\right) = V(\beta_1) + \left(\text{Education}^2 \times V(\beta_3)\right) + \left(2 \times \text{Education} \times Cov(\beta_1, \beta_3)\right).$$

(f) Assuming that $V(\beta_1) = 60000$, $V(\beta_3) = 20000$, and $Cov(\beta_1, \beta_3) = 5000$, the variance we derived in part (e) becomes

$$V\left(\frac{dE(\text{Income})}{d\text{Gender}}\right) = 60000 + 20000\,\text{Education}^2 + 10000\,\text{Education}.$$

The standard errors are given by the square root:

$$SE\left(\frac{dE(\text{Income})}{d\text{Gender}}\right) = \sqrt{60000 + 20000\,\text{Education}^2 + 10000\,\text{Education}}.$$

The standard errors at each education level are listed in the following table

At education level	Standard Error
0	$\sqrt{60000 + 20000(0)^2 + 10000(0)} = 173.21$
1	$\sqrt{60000 + 20000(1)^2 + 10000(1)} = 244.95$
2	$\sqrt{60000 + 20000(2)^2 + 10000(2)} = 360.56$
3	$\sqrt{60000 + 20000(3)^2 + 10000(3)} = 489.90$
4	$\sqrt{60000 + 20000(4)^2 + 10000(4)} = 624.50$
5	$\sqrt{60000 + 20000(5)^2 + 10000(5)} = 761.58$
6	$\sqrt{60000 + 20000(6)^2 + 10000(6)} = 900.00$
7	$\sqrt{60000 + 20000(7)^2 + 10000(7)} = 1039.23$
8	$\sqrt{60000 + 20000(8)^2 + 10000(8)} = 1178.98$
9	$\sqrt{60000 + 20000(9)^2 + 10000(9)} = 1319.09$
10	$\sqrt{60000 + 20000(10)^2 + 10000(10)} = 1459.45$

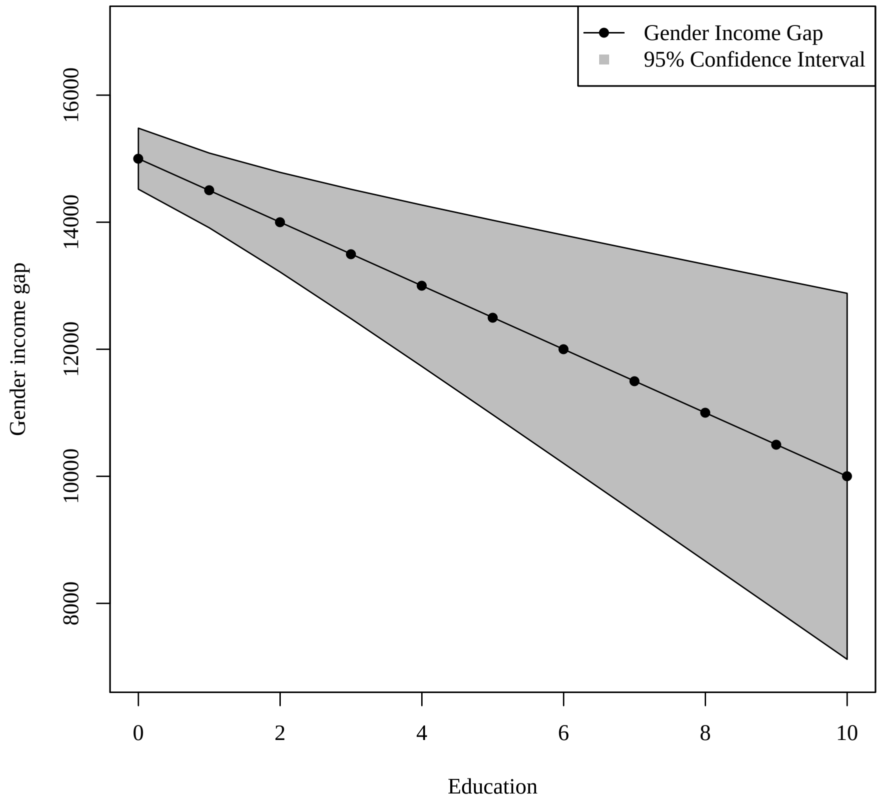
The 95% confidence interval's lower and upper bounds, given by

Lower bound = effect - 1.96 × standard error
Upper bound = effect + 1.96 × standard error,

are:

At education level	Lower bound	Upper bound
0	15000-1.96(173.21)=**14661**	15000+1.96(173.21)=**15339**
1	14500-1.96(244.95)=**14020**	14500+1.96(244.95)=**14980**
2	14000-1.96(360.56)=**13293**	14000+1.96(360.56)=**14707**
3	13500-1.96(489.90)=**12540**	13500+1.96(489.90)=**14460**
4	13000-1.96(624.50)=**11776**	13000+1.96(624.50)=**14224**
5	12500-1.96(761.58)=**11007**	12500+1.96(761.58)=**13993**
6	12000-1.96(900.00)=**10236**	12000+1.96(900.00)=**13764**
7	11500-1.96(1039.23)=**9463**	11500+1.96(1039.23)=**13537**
8	11000-1.96(1178.98)=**8689**	11000+1.96(1178.98)=**13311**
9	10500-1.96(1319.09)=**7915**	10500+1.96(1319.09)=**13085**
10	10000-1.96(1459.45)=**7139**	10000+1.96(1459.45)=**12861**

(g) The graph of the gender income gap against education is again drawn below. This time the lower and upper bounds are drawn on the graph as well, and the entire 95% confidence interval is shaded in grey.



To read this graph: first choose a level of education on the x-axis. Then trace a vertical line up to the middle line. The y-value of the middle line at this point is the average gender income gap among people with the education level you've selected. On the same vertical line, move down to the bottom of the grey area and up to the top of the grey area. The y-values of these two points are the lower and upper bounds of the 95% confidence interval – in statistics we are never certain about any prediction, and here we state that we are 95% confident that the true gender income gap at the given education level is between these two bounds.