Chapter 7: Multivariate Calculus

1. (a) A function with a domain of R^6 and a range of R takes inputs which are ordered sextuplets (6 dimensional numbers) and returns outputs which are unidimensional ("regular" real) numbers. An example of a function of this type is

$$f(x_1, x_2, x_3, x_4, x_5, x_6) = x_1 + x_2 + x_3 + x_4 + x_5 + x_6.$$

This function would map the ordered sextuplet $(1, 2, 3, 4, 5, 6)$ to the real number $1+2+3+4+5+6 = 21$. Incidentally, any linear regression model with one dependent variable and six independent variables is a function that maps $R^6 \to R$, such as

$$f(x_1, x_2, x_3, x_4, x_5, x_6) = \alpha + \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3 + \beta_4 x_4 + \beta_5 x_5 + \beta_6 x_6.$$

(b) A function with a domain of R^3 and a range of R^3 takes inputs which are ordered triplets and returns outputs which are also ordered triplets. An example of a function of this type is

$$f(x_1, x_2, x_3) = \left(x_1 + x_2 + x_3, x_1 x_2 x_3, \frac{x_1 + x_2}{x_3} \right).$$

This function would map the ordered triplet $(1, 2, 3)$ to the ordered triplet $\left(1 + 2 + 3, 1 \times 2 \times 3, \frac{1+2}{3}\right) = (6, 6, 1)$.

(c) A function with a domain of R and a range of R^4 takes inputs which are unidimensional real numbers and returns outputs which are ordered quadruplets. An example of a function of this type is

$$f(x) = (2x, x^2, x - 7, e^x).$$

This function would map 2 to the ordered quadruplet $\left(2(2), 2^2, 2 - 7, e^2\right) = (4, 4, -5, 7.39)$.

2. The first step to solving this problem is simply translating the set-builder notation into English. The domain is

$$\{(x, y) \in \mathbf{R}^2 \,|\, x + y \geq 0\},$$

which translates to the set of all ordered pairs in the set of real-numbered ordered pairs such that the sum of the two numbers in the ordered pair add to at least zero. In other words, we need to find a function that is undefined if $x + y$ is negative.

The range is

$$\{(x, y) \in \mathbf{R} \,|\, f(x, y) \geq 0\},$$

which is the set of (unidimensional) real numbers that are greater than or equal to 0. That means that the function must only be able to output numbers that are zero or positive.

A example of such a function is

$$f(x, y) = \sqrt{x + y}.$$

Since we are taking the square root of $x + y$, that sum must be either positive or zero. And the square root function only returns values that are positive or zero. Therefore this function has the domain and range specified in this problem.

3. (a) First, observe that we cannot simply plug in $x = 3$ and $y = 1$ as that would make the denominator equal zero. But we can use a trick: the numerator,

$$x^2 - xy + 6y^2,$$

is quadratic and can be factored. Factoring a quadratic expression is conceptually trickier when it contains two variables instead of one, but the math is basically the same. Pretend that $y = 1$. Then the expression becomes

$$x^2 - x + 6,$$

and two numbers that add to -1 and multiply to 6 are -3 and 2, so this expression factors to $(x-3)(x+2)$. The more general expression factors to $(x - 3y)(x + 2y)$. So the limit can be rewritten as

$$\lim_{(x,y)\to(3,1)} \frac{(x-3y)(x+2y)}{x-3y}$$
$$= \lim_{(x,y)\to(3,1)} (x + 2y).$$

Since we've cancelled out the denominator, it is now safe to plug in (3,1). The limit equals $3 + 2(1) = 5$.

(b) Since both x and y are approaching infinity, a good strategy is to rewrite the limit so that as many terms as possible become fractions with constants in the numerator and variables in the denominator. We saw in chapter 4 that the limit of such a fraction is 0. This strategy is easier to implement here because the function inside the limit is itself a fraction, and we can multiply or divide the top and bottom of a fraction by the same thing without changing its value. With that in mind, let's divide the top and bottom by x^2y^2. The limit becomes

$$\lim_{(x,y)\to(\infty,\infty)} \frac{2x^2y^2 + 3y^2 - 7x - 15}{7x^2y^2 - 5x^2 + 2y + 10} \frac{\frac{1}{x^2y^2}}{\frac{1}{x^2y^2}}$$
$$= \lim_{(x,y)\to(\infty,\infty)} \frac{2 + \frac{3}{x^2} - \frac{7}{xy^2} - \frac{15}{x^2y^2}}{7 - \frac{5}{y^2} + \frac{2}{x^2y} + \frac{10}{x^2y^2}}$$

Each of these newly created fractions has a constant in the numerator and variables in the denominator. Since both variables go to infinity, these fractions all individually go to zero. The limit is equal to $\frac{2}{7}$.

4. (a) $f(x, y) = (x + y)\sqrt{x - y}$

To find the gradient, we take the first partial derivative of the function with respect to each independent variable, and then arrange these partial derivatives in a vector.

The first partial derivative with respect to x,

$$f_x(x,y) = \frac{\partial}{\partial x}\Big((x+y)\sqrt{x-y}\Big),$$

requires the product rule:

$$f_x(x,y) = \frac{\partial}{\partial x}\Big((x+y)\Big)\sqrt{x-y} + (x+y)\frac{\partial}{\partial x}\Big(\sqrt{x-y}\Big).$$

This expression contains two partial derivatives. The first is

$$\frac{\partial}{\partial x}\Big((x+y)\Big) = 1.$$

Remember that we are taking the partial derivative with respect to x, meaning that we treat x as the variable and y as a constant. Here the derivative of x is 1 and the derivative of y, treated as constant, is 0. Next, the second partial derivative is

$$\frac{\partial}{\partial x}\Big(\sqrt{x-y}\Big) = \frac{1}{2\sqrt{x-y}},$$

where again y does not enter into the calculation because it is being treated as a constant. Substituting these two partial derivatives into the overall partial derivative gives us

$$f_x(x,y) = \sqrt{x-y} + \frac{x+y}{2\sqrt{x-y}}.$$

The first partial derivative with respect to y,

$$f_y(x,y) = \frac{\partial}{\partial y}\Big((x+y)\sqrt{x-y}\Big),$$

requires nearly the same calculation as the partial derivative with respect to x. The only difference is that the chain rule implies that we multiply the derivative of the square root by -1:

$$f_y(x,y) = \sqrt{x-y} - \frac{x+y}{2\sqrt{x-y}}.$$

Therefore the gradient is

$$\nabla f(x,y) = \begin{bmatrix} \sqrt{x-y} + \frac{x+y}{2\sqrt{x-y}} \\ \\ \sqrt{x-y} - \frac{x+y}{2\sqrt{x-y}} \end{bmatrix}.$$

(b) $g(x,y) = e^{x^2+y^2-2x+5y+7}$

Remember from chapter 4 that the derivative of e^x is simply e^x again, and that the chain rule tells us that if $f(x) = e^{g(x)}$, then

$$f'(x) = e^{g(x)}g'(x).$$

With that in mind, the partial derivative with respect to x is

$$g_x(x,y) = e^{x^2+y^2-2x+5y+7}\left(\frac{\partial}{\partial x}(x^2+y^2-2x+5y+7)\right),$$

and the partial derivative with respect to y is

$$g_y(x,y) = e^{x^2+y^2-2x+5y+7}\left(\frac{\partial}{\partial y}(x^2+y^2-2x+5y+7)\right).$$

All we have to do to find the gradient is to find

$$\frac{\partial}{\partial x}(x^2+y^2-2x+5y+7) \quad \text{and} \quad \frac{\partial}{\partial y}(x^2+y^2-2x+5y+7)$$

and substitute them into the overall partial derivatives. Treating x as a variable and y as a constant shows us that

$$\frac{\partial}{\partial x}(x^2+y^2-2x+5y+7) = 2x-2.$$

Treating y as a variable and x as a constant shows us that

$$\frac{\partial}{\partial y}(x^2+y^2-2x+5y+7) = 2y+5.$$

Therefore the gradient is

$$\nabla g(x,y) = \begin{bmatrix} (2x-2)e^{x^2+y^2-2x+5y+7} \\ \\ (2y+5)e^{x^2+y^2-2x+5y+7} \end{bmatrix}.$$

(c) $h(x,y) = \ln(x+\sqrt{y})$

Remember from chapter 4 that the derivative of $\ln(x)$ is $\frac{1}{x}$ again, and that the chain rule tells us that if $f(x) = \ln\left(g(x)\right)$, then

$$f'(x) = \frac{1}{g(x)}g'(x) = \frac{g'(x)}{g(x)}.$$

With that in mind, the partial derivative with respect to x is

$$h_x(x,y) = \frac{1}{\ln(x+\sqrt{y})}\left(\frac{\partial}{\partial x}(x+\sqrt{y})\right),$$

and the partial derivative with respect to y is

$$h_y(x,y) = \frac{1}{\ln(x+\sqrt{y})}\left(\frac{\partial}{\partial y}(x+\sqrt{y})\right),$$

All we have to do to find the gradient is to find

$$\frac{\partial}{\partial x}(x+\sqrt{y}) \quad \text{and} \quad \frac{\partial}{\partial y}(x+\sqrt{y})$$

and substitute them into the overall partial derivatives. Treating x as a variable and y as a constant shows us that

$$\frac{\partial}{\partial x}(x+\sqrt{y}) = 1.$$

Treating y as a variable and x as a constant shows us that

$$\frac{\partial}{\partial y}(x+\sqrt{y}) = \frac{1}{2\sqrt{y}}.$$

Therefore the gradient is

$$\nabla h(x,y) = \begin{bmatrix} \frac{1}{x+\sqrt{y}} \\ \\ \frac{1}{2\sqrt{y}(x+\sqrt{y})} \end{bmatrix}.$$

(d) $j(x,y) = \dfrac{x^2+y^2}{x^3-4xy-y^2}$

Both partial derivatives require the quotient rule. The partial derivative with respect to x is

$$j_x(x,y) = \frac{(x^3-4xy-y^2)\frac{\partial}{\partial x}(x^2+y^2) - (x^2+y^2)\frac{\partial}{\partial x}(x^3-4xy-y^2)}{(x^3-4xy-y^2)^2},$$

and the partial derivative with respect to y is

$$j_y(x,y) = \frac{(x^3-4xy-y^2)\frac{\partial}{\partial y}(x^2+y^2) - (x^2+y^2)\frac{\partial}{\partial y}(x^3-4xy-y^2)}{(x^3-4xy-y^2)^2}.$$

All we have to do to find these partial derivatives is calculate the smaller derivatives inside these expressions. These derivatives are

$$\frac{\partial}{\partial x}(x^2+y^2) = 2x \quad , \quad \frac{\partial}{\partial x}(x^3-4xy-y^2) = 3x^2-4y,$$

$$\frac{\partial}{\partial y}(x^2+y^2) = 2y, \quad \text{and} \quad \frac{\partial}{\partial y}(x^3-4xy-y^2) = 2y-4x.$$

Substituting into the overall partial derivatives, the gradient is

$$\nabla j(x,y) = \begin{bmatrix} \frac{2x(x^3-4xy-y^2)-(3x^2-4y)(x^2+y^2)}{(x^3-4xy-y^2)^2} \\ \\ \frac{2y(x^3-4xy-y^2)-(2y-4x)(x^2+y^2)}{(x^3-4xy-y^2)^2} \end{bmatrix}.$$

(e) $k(x,y,z) = -4x^5y^3z^2 - 3y^2z^4 + 7xz^3 + 10y - 9z + 9$

Since there are three independent variables, we have to find three partial derivatives: one with respect to x, y, and z.

To find the partial derivative with respect to x, we treat y and z as constant. Then finding the derivative is straightforward:

$$k_x(x,y,z) = \frac{\partial}{\partial x}\Big(-4x^5y^3z^2 - 3y^2z^4 + 7xz^3 + 10y - 9z + 9\Big) = -20x^4y^3z^2 + 7z^3.$$

Similarly, the partial derivative with respect to y is

$$k_y(x,y,z) = \frac{\partial}{\partial y}\Big(-4x^5y^3z^2 - 3y^2z^4 + 7xz^3 + 10y - 9z + 9\Big) = -12x^5y^2z^2 - 6yz^4 + 10,$$

and the partial derivative with respect to z is

$$k_z(x,y,z) = \frac{\partial}{\partial z}\Big(-4x^5y^3z^2 - 3y^2z^4 + 7xz^3 + 10y - 9z + 9\Big) = -8x^5y^3z - 12y^2z^3 + 21xz^2 - 9.$$

The gradient is

$$\nabla k(x,y,z) = \begin{bmatrix} -20x^4y^3z^2 + 7z^3 \\ \\ -12x^5y^2z^2 - 6yz^4 + 10 \\ \\ -8x^5y^3z - 12y^2z^3 + 21xz^2 - 9 \end{bmatrix}.$$

(f) $l(x,y,z) = \dfrac{x^2 - y^2 + z^2}{\ln(x)}$

The partial derivative with respect to x requires the quotient rule since x appears in the numerator and the denominator:

$$l_x(x,y,z) = \frac{\partial}{\partial x}\left(\frac{x^2 - y^2 + z^2}{\ln(x)}\right) = \frac{\ln(x)\frac{\partial}{\partial x}(x^2 - y^2 + z^2) - (x^2 - y^2 + z^2)\frac{\partial}{\partial x}(\ln(x))}{\ln(x)^2}$$

$$= \frac{2x\ln(x) - \frac{x^2-y^2+z^2}{x}}{\ln(x)^2}$$

$$= \frac{2x\ln(x)}{\ln(x)^2} - \frac{\frac{x^2-y^2+z^2}{x}}{\ln(x)^2}$$

$$= \frac{2x}{\ln(x)} - \frac{x^2 - y^2 + z^2}{x\ln(x)^2}.$$

The partial derivatives of y and z *do not* require the quotient rule because neither y nor z appear in the denominator. We can treat the denominator as a constant factor that can be brought outside of these derivatives. The partial derivative with respect to y is

$$l_y(x,y,z) = \frac{\partial}{\partial y}\left(\frac{x^2 - y^2 + z^2}{\ln(x)}\right)$$

$$= \frac{1}{\ln(x)}\frac{\partial}{\partial y}(x^2 - y^2 + z^2)$$

$$= \frac{-2y}{\ln(x)},$$

and the partial derivative with respect to z is

$$l_z(x,y,z) = \frac{\partial}{\partial z}\left(\frac{x^2 - y^2 + z^2}{\ln(x)}\right)$$

$$= \frac{1}{\ln(x)}\frac{\partial}{\partial z}(x^2 - y^2 + z^2)$$

$$= \frac{2z}{\ln(x)}.$$

The gradient is

$$\nabla l(x,y,z) = \begin{bmatrix} \frac{2x}{\ln(x)} - \frac{x^2-y^2+z^2}{x\ln(x)^2} \\ \\ \frac{-2y}{\ln(x)} \\ \\ \frac{2z}{\ln(x)} \end{bmatrix}.$$

5. A gradient is a vector of the first partial derivatives. The partial derivative of $f(x,y)$ with respect to x is

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\Big((x^2 - y^2)\ln(x+y)\Big)$$

$$= (x^2 - y^2)\frac{\partial}{\partial x}\ln(x+y) + \ln(x+y)\frac{\partial}{\partial x}(x^2 - y^2)$$

$$= \frac{x^2 - y^2}{x + y} + 2x \ln(x + y)$$

$$= \frac{(x + y)(x - y)}{x + y} + 2x \ln(x + y)$$

$$= x - y + 2x \ln(x + y).$$

The partial derivative of $f(x, y)$ with respect to y is

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\Big((x^2 - y^2) \ln(x + y)\Big)$$

$$= (x^2 - y^2)\frac{\partial}{\partial y} \ln(x + y) + \ln(x + y)\frac{\partial}{\partial y}(x^2 - y^2)$$

$$= \frac{x^2 - y^2}{x + y} - 2y \ln(x + y)$$

$$= \frac{(x + y)(x - y)}{x + y} - 2y \ln(x + y)$$

$$= x - y - 2y \ln(x + y).$$

Therefore the gradient of $f(x, y)$ is

$$\nabla f(x, y) = \begin{bmatrix} x - y + 2x \ln(x + y) \\ \\ x - y - 2y \ln(x + y) \end{bmatrix}.$$

The Hessian is the matrix of second partial derivatives. We have to calculate the partial derivative with respect to x and x again, the partial derivative with respect to y and y again, and the partial derivative with respect to x and then y which must be equal to the partial derivative with respect to y then x. First, we calculate the partial derivative with respect to x and x again:

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial^2}{\partial x^2}\Big((x^2 - y^2) \ln(x + y)\Big)$$

$$= \frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}\Big((x^2 - y^2) \ln(x + y)\Big)\right)$$

$$= \frac{\partial}{\partial x}\Big(x - y + 2x \ln(x + y)\Big)$$

$$= 1 + 2 \ln(x + y) + \frac{2x}{x + y}$$

$$= \frac{-(x + y)}{x + y} + 2 \ln(x + y) + \frac{2x}{x + y}$$

$$= \frac{-x - y + 2x}{x + y} - +2 \ln(x + y)$$

$$= \frac{x - y}{x + y} + 2 \ln(x + y).$$

Next, we calculate the partial derivative with respect to y and y again:

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial^2}{\partial y^2}\Big((x^2 - y^2) \ln(x + y)\Big)$$

$$= \frac{\partial}{\partial y}\left(\frac{\partial}{\partial y}\Big((x^2 - y^2) \ln(x + y)\Big)\right)$$

$$= \frac{\partial}{\partial y}\Big(x - y - 2y \ln(x + y)\Big)$$

$$= -1 - 2\ln(x+y) - \frac{2y}{x+y}$$

$$= \frac{-(x+y)}{x+y} - 2\ln(x+y) + \frac{2x}{x+y}$$

$$= \frac{-x-y+2x}{x+y} - 2\ln(x+y)$$

$$= \frac{x-y}{x+y} - 2\ln(x+y).$$

Finally, the partial derivative with respect to x then y is

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2}{\partial x \partial y}\Big((x^2-y^2)\ln(x+y)\Big)$$

$$= \frac{\partial}{\partial y}\left(\frac{\partial}{\partial x}\Big((x^2-y^2)\ln(x+y)\Big)\right)$$

$$= \frac{\partial}{\partial y}\Big(x - y + 2x\ln(x+y)\Big)$$

$$= -1 + \frac{2x}{x+y}$$

$$= \frac{-(x+y)}{x+y} + \frac{2x}{x+y}$$

$$= \frac{-x-y+2x}{x+y}$$

$$= \frac{x-y}{x+y}.$$

So the Hessian of $f(x,y)$ is

$$H\Big(f(x,y)\Big) = \begin{bmatrix} \frac{x-y}{x+y} + 2\ln(x+y) & \frac{x-y}{x+y} \\ \frac{x-y}{x+y} & \frac{x-y}{x+y} - 2\ln(x+y) \end{bmatrix}.$$

6. (a) The first partial derivative of the function with respect to x is

$$\frac{\partial f(x,y)}{\partial x} = \frac{\partial}{\partial x}\Big(-x^2 + xy - y^2 + 2x + y\Big) = -2x + y + 2.$$

The first partial derivative with respect to y is

$$\frac{\partial f(x,y)}{\partial y} = \frac{\partial}{\partial y}\Big(-x^2 + xy - y^2 + 2x + y\Big) = x - 2y + 1.$$

So the gradient of $f(x,y)$ is

$$\nabla f(x,y) = \begin{bmatrix} -2x + y + 2 \\ x - 2y + 1 \end{bmatrix}.$$

(b) We now have to solve the following system of equations:

$$\begin{cases} -2x + y + 2 = 0, \\ x - 2y + 1 = 0. \end{cases}$$

We can solve this system by solving the top equation for y,

$$y = 2x - 2,$$

and plugging in for y in the second equation:

$$\begin{aligned} x - 2(2x - 2) + 1 &= 0, \\ x - 4x + 4 + 1 &= 0, \\ -3x + 5 &= 0, \\ 3x &= 5, \\ x &= \frac{5}{3}. \end{aligned}$$

Then plugging the solution for x back into the first equation solved for y, we get

$$y = 2(5/3) - 2 = 10/3 - 6/3 = \frac{4}{3}.$$

So the ordered pair $(x, y) = \left(\frac{5}{3}, \frac{4}{3}\right)$ is a critical point for the function. Since we obtained only one possible value of x, and this value implied only one value of y, that means that $(x, y) = \left(\frac{5}{3}, \frac{4}{3}\right)$ is the only critical point.

(c) First, the second partial derivative with respect to x and x again is

$$\frac{\partial f(x,y)}{\partial x} = \frac{\partial}{\partial x}\Big(- 2x + y + 2\Big) = -2$$

The second partial derivative with respect to y and y again is

$$\frac{\partial f(x,y)}{\partial y} = \frac{\partial}{\partial y}\Big(x - 2y + 1\Big) = -2.$$

Finally, the second partial derivative with respect to x then y (or y then x) is

$$\frac{\partial f(x,y)}{\partial y} = \frac{\partial}{\partial y}\Big(- 2x + y + 2\Big) = 1.$$

So the Hessian is

$$H\Big(f(x,y)\Big) = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}.$$

(d) First we check whether

$$f_{xx}(-1,-4)f_{yy}(-1,-4) - f_{xy}(-1,-4)^2 > 0,$$

$$
\begin{aligned}
(-2)(-2)-(1)^2 &> 0,\\
4-1 &> 0,\\
3 &> 0,
\end{aligned}
$$

so the first condition is met. Next we check the sign of

$$f_{xx}(-1,-4)+f_{yy}(-1,-4) = -2+-2 = -4 < 0.$$

So the critical point represents a local maximum.

7. The problem is to maximize the function

$$f(x,y) = 150x^{1/3}y^{2/3}$$

subject to the constraint that

$$300x + 500y = 100000.$$

In order to apply the method of Lagrange multipliers, we first find the gradient of $f(x,y)$. The partial derivative with respect to x is

$$
\begin{aligned}
f_x(x,y) &= \frac{\partial}{\partial x}(150x^{1/3}y^{2/3})\\
&= 150y^{2/3}\frac{\partial}{\partial x}(x^{1/3})\\
&= 150y^{2/3}\left(\frac{1}{3}x^{-2/3}\right)\\
&= \frac{50y^{2/3}}{x^{2/3}}.
\end{aligned}
$$

The partial derivative with respect to y is

$$
\begin{aligned}
f_y(x,y) &= \frac{\partial}{\partial y}(150x^{1/3}y^{2/3})\\
&= 150x^{1/3}\frac{\partial}{\partial y}(y^{2/3})\\
&= 150x^{1/3}\left(\frac{2}{3}y^{-1/3}\right)\\
&= \frac{100x^{1/3}}{y^{1/3}}.
\end{aligned}
$$

The gradient is

$$\nabla f(x,y) = \begin{bmatrix} \frac{50y^{2/3}}{x^{2/3}} \\ \\ \frac{100x^{1/3}}{y^{1/3}} \end{bmatrix}.$$

Note that there are no critical points for this function. The point (0,0) does not make the gradient's elements equal 0 because it places 0 in the denominator of each partial derivative.

Next, we write the constraint function, replacing the constant 100,000 with a general function name $g(x,y)$,

$$g(x,y) = 300x + 500y.$$

The partial derivative of $g(x, y)$ with respect to x is 300, and the partial derivative of $g(x, y)$ with respect to y is 500. So the gradient of $g(x, y)$ is

$$\nabla g(x, y) = \begin{bmatrix} 300 \\ 500 \end{bmatrix}.$$

Next we create a system of three equations by setting $f(x, y) = \lambda g(x, y)$, and including the constraint as well:

$$\begin{cases} \frac{50y^{2/3}}{x^{2/3}} = 300\lambda, \\ \\ \frac{100x^{1/3}}{y^{1/3}} = 500\lambda, \\ \\ 300x + 500y = 100000. \end{cases}$$

To solve this system, let's first solve the first two equations for λ then set them equal to each other. Dividing both sides by 300, the first equation can be rewritten as

$$\frac{y^{2/3}}{6x^{2/3}} = \lambda,$$

and dividing both sides by 500, the second equation can be written as

$$\frac{x^{1/3}}{5y^{1/3}} = \lambda.$$

These two equations imply that

$$\begin{aligned} \frac{y^{2/3}}{6x^{2/3}} &= \frac{x^{1/3}}{5y^{1/3}}, \\ (y^{2/3})(5y^{1/3}) &= (x^{1/3})(6x^{2/3}), \\ 5y &= 6x, \\ y &= \frac{6}{5}x. \end{aligned}$$

Finally we substitute for y in the last equation:

$$\begin{aligned} 300x + 500\left(\frac{6}{5}x\right) &= 100000, \\ 300x + 600x &= 100000, \\ 900x &= 100000, \\ x &= \frac{100000}{900} = 111.11. \end{aligned}$$

Given this value of x, we can calculate y:

$$y = \frac{6}{5}(111.11) = 133.33.$$

We have no critical points to compare the point (111.11, 133.33) to, and it is our only optimum on the line $300x + 500y = 100000$. The value of the Cobb-Douglas production function at this point is

$$f(111.11, 133.33) = 150(111.11)^{1/3}(133.33)^{2/3} = 18820.34.$$

Any other point on this line produces a lower value of $f(x, y)$ than $f(111.11, 133.33)$. Consider, for example, the case in which $x = 50$. According to the constraint, the value of y must be

$$\begin{aligned} 300(50) + 500y &= 100000, \\ 15000 + 500y &= 100000, \\ 500y &= 85000, \end{aligned}$$

$$y = 170.$$

The value of the Cobb-Douglas production function at the point (50,170) is

$$f(50, 170) = 150(50)^{1/3}(170)^{2/3} = 16958.23.$$

Therefore the firm maximizes its profits by spending its resources on 111.11 units of labor and 133.33 units of capital.

8. (a) The problem asks us to solve the definite double integral

$$\int_0^3 \int_1^5 3x^2 - 3y^2 - 2xy \; dy \; dx.$$

Because neither set of bounds depends on a variable, these bounds can be considered in either order. This integral is equal to the one in which the order of x and y are reversed:

$$\int_1^5 \int_0^3 3x^2 - 3y^2 - 2xy \; dx \; dy.$$

To solve this double integral, we start by solving the innermost integral:

$$\int_0^3 \left(\int_1^5 3x^2 - 3y^2 - 2xy \; dy \right) dx$$

$$= \int_0^3 \left(3x^2y - y^3 - xy^2 \Big|_1^5 \right) dx$$

$$= \int_0^3 \left([3x^2(5) - (5)^3 - x(5)^2] - [3x^2(1) - (1)^3 - x(1)^2] \right) dx$$

$$= \int_0^3 \left([15x^2 - 125 - 25x] - [3x^2 - 1 - x] \right) dx$$

$$= \int_0^3 12x^2 - 124 - 24x \; dx.$$

We now have a single definite integral to solve:

$$\int_0^3 12x^2 - 124 - 24x \; dx = 4x^3 - 124x - 12x^2 \Big|_0^3$$

$$= 4(3)^3 - 124(3) - 12(3)^2 - (4(0)^3 - 124(0) - 12(0)^2) = -372.$$

(b) The problem asks us to solve

$$\int_2^4 \int_1^{e^x} \frac{x}{y} \; dy \; dx.$$

Note that, unlike part (a), here we have no choice in the order of the bounds. We must work with y before x because the bounds of y themselves contain a function of x. We solve the inner-integral first:

$$\int_2^4 \left(\int_1^{e^x} \frac{x}{y} \; dy \right) dx$$

In that inner-integral, we treat y as the variable and x as a constant. Since x is a constant factor, we can bring it outside that integral:

$$\int_2^4 x\left(\int_1^{e^x} \frac{1}{y}\, dy\right)\, dx$$

$$= \int_2^4 x\left(\ln(y)\Big|_1^{e^x}\right)\, dx^1$$

$$= \int_2^4 x\left(\ln(e^x) - \ln(1)\right)\, dx$$

$$= \int_2^4 x\left(x - 0\right)\, dx$$

$$= \int_2^4 x^2\, dx = \frac{x^3}{3}\Big|_2^4$$

$$\frac{4^3}{3} - \frac{2^3}{3} = \frac{64-8}{3} = \frac{56}{3}.$$

(c) The problem asks us to solve

$$\int_0^1 \int_0^{3x^2} \sqrt{x^2+y}\, dy\, dx.$$

Note that, like part (b), we must work with y before x because the bounds of y themselves contain a function of x. We solve the inner-integral first:

$$\int_0^1 \left(\int_0^{3x^2} \sqrt{x^2+y}\, dy\right)\, dx.$$

$$= \int_0^1 \left(\int_0^{3x^2} (x^2+y)^{1/2}\, dy\right)\, dx.$$

We need to employ u-substitution to solve the inner-integral. Let $u = x^2 + y$. Then $\frac{du}{dy} = 1$, and so $du = dy$. We also change the bounds:

$$u(0) = x^2 + 0 = x^2, \quad \text{and} \quad u(x^2) = x^2 + (3x^2) = 4x^2.$$

Substituting, the integral becomes:

$$\int_0^1 \left(\int_{x^2}^{4x^2} u^{1/2}\, du\right)\, dx$$

$$= \int_0^1 \left(\frac{2}{3}u^{3/2}\Big|_{x^2}^{4x^2}\right)\, dx$$

$$= \frac{2}{3}\int_0^1 \left((4x^2)^{3/2} - (x^2)^{3/2}\right)\, dx$$

$$= \frac{2}{3}\int_0^1 \left((4^{3/2}(x^2)^{3/2} - (x^2)^{3/2}\right)\, dx$$

$$= \frac{2}{3}\int_0^1 \left((4^{3/2} - 1)(x^2)^{3/2}\right)\, dx$$

[1] Technically, the anti-derivative of $\frac{1}{y}$ is $\ln(|y|)$. Here we can omit the absolute value because all of the numbers within the range of the inner-integral are positive.

$$= \frac{2}{3}\int_0^1 7x^3\ dx$$

$$= \frac{14}{3}\int_0^1 x^3\ dx = \frac{14}{3}\left(\frac{x^4}{4}\bigg|_0^1\right)$$

$$= \frac{14}{3}\left(\frac{1}{4} - \frac{0}{4}\right) = \frac{7}{6}.$$

(d) The problem asks us to solve

$$\int_1^{e^4}\int_1^{e^5} \frac{\ln(x) + \ln(y)}{xy}\ dx\ dy.$$

Like part (a), neither set of bounds depends on the other variable, so we can consider these bounds in any order. Let's consider the bounds of x first. Note that the y in the denominator can be considered to be a constant factor that can be brought outside of this integral:

$$\int_1^{e^4} \frac{1}{y}\left(\int_1^{e^5} \frac{\ln(x) + \ln(y)}{x}\ dx\right)\ dy.$$

We need to use u-substitution to solve the inner-integral. Let $u = \ln(x) + \ln(y)$. Remember that in this inner-integral, we are treating x as the variable and y as constant, so we take the partial derivative of u with respect to x,

$$\frac{du}{dx} = \frac{1}{x}, \qquad du = \frac{1}{x}\ dx.$$

Notice that the factor $\frac{1}{x}$ will cancel out of the integrand. We also change the bounds:

$$u(1) = \ln(1) + \ln(y) = \ln(y), \quad \text{and} \quad u(e^5) = \ln(e^5) + \ln(y) = 5 + \ln(y).$$

Substituting, the integral becomes

$$\int_1^{e^4} \frac{1}{y}\left(\int_{\ln(y)}^{5+\ln(y)} u\ du\right)\ dy$$

$$= \int_1^{e^4} \frac{1}{y}\left(\frac{u^2}{2}\bigg|_{\ln(y)}^{5+\ln(y)}\right)\ dy$$

$$= \int_1^{e^4} \frac{1}{2y}\left((5 + \ln(y))^2 - \ln(y)^2\right)\ dy$$

$$= \int_1^{e^4} \frac{1}{2y}\left(25 + 10\ln(y) + \ln(y)^2 - \ln(y)^2\right)\ dy$$

$$= \int_1^{e^4} \frac{1}{2y}\left(25 + 10\ln(y)\right)\ dy$$

$$= \frac{1}{2}\int_1^{e^4} \frac{25 + 10\ln(y)}{y}\ dy.$$

Now we have to employ u-substitution again to solve this remaining integral, which treats y as the variable. Let $u = 25 + 10\ln(y)$. Then

$$\frac{du}{dy} = \frac{10}{y}, \qquad \frac{1}{10}du = \frac{1}{y}\ dy.$$

Now the factor $\frac{1}{y}$ cancels out of the integrand. We replace the bounds:

$$u(1) = 25 + \ln(1) = 25, \quad \text{and} \quad u(e^4) = 25 + \ln(e^4) = 29.$$

Substituting, the integral becomes

$$\begin{aligned}
&\frac{1}{2}\int_{25}^{29} u \left(\frac{1}{10}du\right) \\
&= \frac{1}{20}\frac{u^2}{2}\Big|_{25}^{29} \\
&= \frac{29^2 - 25^2}{40} = 5.4.
\end{aligned}$$

9. (a) In order to demonstrate that a multivariate function is a joint PDF, we have to show that the function is nonnegative for all ordered pairs in the domain, and that the multiple integral over the domain evaluates to 1. First, note that the domain only contains nonnegative values of x and y, so

$$f(x, y) = \frac{6}{7}\left(x^2 + \frac{xy}{2}\right)$$

must also be nonnegative since it contains no subtraction or multiplication by negative numbers. Next we have to demonstrate that the integral

$$\int_0^1 \int_0^2 \frac{6}{7}\left(x^2 + \frac{xy}{2}\right) \, dy \, dx = 1.$$

The integral is solved using the following steps:

$$\begin{aligned}
&= \frac{6}{7}\int_0^1 \int_0^2 x^2 + \frac{xy}{2} \, dy \, dx \\
&= \frac{6}{7}\int_0^1 \left(\int_0^2 x^2 + \frac{xy}{2} \, dy\right) dx \\
&= \frac{6}{7}\int_0^1 \left(x^2 y + \frac{xy^2}{4}\Big|_0^2\right) dx \\
&= \frac{6}{7}\int_0^1 \left(2x^2 + \frac{4x}{4}\right) - \left(0x^2 + \frac{0x}{4}\right) dx \\
&= \frac{6}{7}\int_0^1 2x^2 + x \, dx \\
&= \frac{6}{7}\left(\frac{2}{3}x^3 + \frac{x^2}{2}\Big|_0^1\right) \\
&= \frac{6}{7}\left(\left(\frac{2}{3}1^3 + \frac{1^2}{2}\right) - \left(\frac{4}{3}0^3 + \frac{0^2}{2}\right)\right) \\
&= \frac{6}{7}\left(\frac{2}{3} + \frac{1}{2}\right) \\
&= \frac{6}{7}\left(\frac{4}{6} + \frac{3}{6}\right) \\
&= \frac{6}{7}\left(\frac{7}{6}\right) = 1.
\end{aligned}$$

So the function is indeed a joint PDF.

(b) Now that we've demonstrated that $f(x, y)$ is a joint PDF, we can calculate the probability that x and y fall within certain bounds by plugging these new bounds into the multivariate definite integral and finding the volume under the curve. In this case, the problem is to solve

$$P(0 < x < 0.5, 0 < y < x) = \int_0^{.5} \int_0^{x} \frac{6}{7}\left(x^2 + \frac{xy}{2}\right) dy\ dx.$$

The integral is solved using the following steps:

$$= \frac{6}{7} \int_0^{.5} \int_0^{x} x^2 + \frac{xy}{2}\ dy\ dx$$

$$= \frac{6}{7} \int_0^{.5} \left(\int_0^{x} x^2 + \frac{xy}{2}\ dy \right) dx$$

$$= \frac{6}{7} \int_0^{.5} \left(x^2 y + \frac{xy^2}{4} \Big|_0^x \right) dx$$

$$= \frac{6}{7} \int_0^{.5} \left(x^3 + \frac{x^3}{4} \right) - \left(0x^2 + \frac{0x}{4} \right) dx$$

$$= \frac{6}{7} \int_0^{.5} \left(\frac{5}{4} x^3 \right) dx$$

$$= \frac{15}{14} \int_0^{.5} x^3\ dx$$

$$= \frac{15}{14} \frac{x^4}{4} \Big|_0^{.5}$$

$$= \frac{15}{14} \left(\frac{.5^4}{4} - \frac{0^4}{4} \right)$$

$$= \frac{15}{14} \times \frac{1}{64} = \frac{15}{896} = 0.017.$$

(c) To find the marginal distribution of x, we integrate the joint PDF over the domain of y:

$$f_x(x) = \int_{-\infty}^{\infty} f(x, y)\ dy$$

$$= \int_0^{2} \frac{6}{7}\left(x^2 + \frac{xy}{2}\right) dy$$

$$= \frac{6}{7} \int_0^{2} x^2 + \frac{xy}{2}\ dy$$

$$= \frac{6}{7} \left(x^2 y + \frac{xy^2}{4} \Big|_0^2 \right)$$

$$= \frac{6}{7} \left(x^2(2) + \frac{x(2)^2}{4} - x^2(0) - \frac{x(0)^2}{4} \Big|_0^2 \right)$$

$$= \frac{6}{7} \left(2x^2 + x \right)$$

$$= \frac{12x^2 + 6x}{7}.$$

(d) We repeat the calculations we conducted for part (c), but this time we integrate the joint PDF over the domain of x:

$$
\begin{aligned}
f_y(y) &= \int_{-\infty}^{\infty} f(x,y)\ dx \\
&= \int_0^1 \frac{6}{7}\left(x^2 + \frac{xy}{2}\right)\ dx \\
&= \frac{6}{7}\left(\frac{x^3}{3} + \frac{x^2 y}{4}\bigg|_0^1\right) \\
&= \frac{6}{7}\left(\frac{(1)^3}{3} + \frac{(1)^2 y}{4} - \frac{(0)^3}{3} - \frac{(0)^2 y}{4}\right) \\
&= \frac{6}{7}\left(\frac{1}{3} + \frac{y}{4}\right) \\
&= \frac{4+3y}{14}.
\end{aligned}
$$

(e) x and y are independent if and only if the product of their marginal distributions equals the joint PDF. The product of the marginal distributions is

$$
\begin{aligned}
&f_x(x)f_y(y) \\
&= \left(\frac{12x^2+6x}{7}\right)\left(\frac{4+3y}{14}\right) \\
&= \frac{(12x^2+6x)(4+3y)}{98} \\
&= \frac{48x^2+24x+36x^2y+18xy}{98},
\end{aligned}
$$

which does not equal the joint PDF $\frac{6}{7}\left(x^2+\frac{xy}{2}\right)$. To see this explicitly, consider the point (1,0) which is in the domain of the joint PDF. At this point, the product of the marginal distributions is

$$
f_x(1)f_y(0) = \frac{48(1)^2+24(1)+36(1)^2(0)+18(1)(0)}{98} = \frac{48+24}{98} = 0.735.
$$

And the joint PDF at (1,0) is

$$
f(1,0) = \frac{6}{7}\left((1)^2 + \frac{(1)(0)}{2}\right) = \frac{6}{7} = 0.857.
$$

Since there are instances in which the product of marginal distributions does not equal the joint PDF, these variables are not independent.

(f) The conditional distribution of x given y is given by the quotient of the joint PDF and the marginal distribution of y:

$$
f_{x|y}(x,y) = \frac{f(x,y)}{f_y(x,y)}.
$$

Since we know the joint PDF, and we've already derived the marginal distribution of y in part (d), we do not need to take any more integrals – we simply plug these two known functions into the above formula:

$$f_{x|y}(x,y) = \frac{\frac{6}{7}\left(x^2 + \frac{xy}{2}\right)}{\frac{4+3y}{14}}$$

$$= \frac{6}{7}\left(x^2 + \frac{xy}{2}\right) \times \frac{14}{4+3y}$$

$$= \frac{12\left(x^2 + \frac{xy}{2}\right)}{4+3y}$$

$$= \frac{12x^2 + 6xy}{4+3y}.$$

(g) The conditional distribution of y given x is given by the quotient of the joint PDF and the marginal distribution of x:

$$f_{y|x}(x,y) = \frac{f(x,y)}{f_x(x,y)}.$$

Since we know the joint PDF, and we've already derived the marginal distribution of x in part (c), we do not need to take any more integrals – we simply plug these two known functions into the above formula:

$$f_{y|x}(x,y) = \frac{\frac{6}{7}\left(x^2 + \frac{xy}{2}\right)}{\frac{12x^2+6x}{7}}$$

$$= \frac{6}{7}\left(x^2 + \frac{xy}{2}\right) \times \frac{7}{12x^2+6x}$$

$$= \frac{6\left(x^2 + \frac{xy}{2}\right)}{12x^2+6x}$$

$$= \frac{x^2 + \frac{xy}{2}}{2x^2+x}$$

$$= \frac{2x^2 + xy}{4x^2+2x}$$

$$= \frac{2x+y}{4x+2}.$$

(h) The expected value of x is given by the following formula,

$$E(x) = \int_{-\infty}^{\infty} x\, f_x(x)\ dx,$$

where the infinite bounds stand in for whatever the domain of x happens to be, and $f_x(x)$ is the marginal distribution of x. We found the marginal distribution of x in part (c), so in this case the expected value is

$$E(x) = \int_0^1 x\left(\frac{12x^2+6x}{7}\right)\ dx$$

$$= \frac{6}{7}\int_0^1 2x^3 + x^2\ dx$$

$$= \frac{6}{7}\left(\frac{x^4}{2} + \frac{x^3}{3}\Big|_0^1\right)$$

$$= \frac{6}{7}\left(\frac{1^4}{2} + \frac{1^3}{3} - \frac{0^4}{2} - \frac{0^3}{3}\right)$$

$$= \frac{6}{7}\left(\frac{1}{2} + \frac{1}{3}\right)$$

$$= \frac{6}{7} \times \frac{5}{6} = \frac{5}{7} = .714.$$

(i) The expected value of y is given by

$$E(y) = \int_{-\infty}^{\infty} y\, f_y(y)\ dy,$$

where the infinite bounds stand in for whatever the domain of y happens to be, and $f_y(y)$ is the marginal distribution of y. We found the marginal distribution of y in part (d), so in this case the expected value is

$$E(y) = \int_0^2 y\left(\frac{4+3y}{14}\right)\ dy$$

$$= \frac{1}{14}\int_0^2 4y + 3y^2\ dy$$

$$= \frac{1}{14}\left(2y^2 + y^3\Big|_0^2\right)$$

$$= \frac{1}{14}\left(2(2)^2 + (2)^3 - 2(0)^2 - (0)^3\right)$$

$$= \frac{1}{14}(8+8) = \frac{16}{14} = \frac{8}{7} = 1.14.$$

(j) The variance of x is given by

$$V(x) = \int_{-\infty}^{\infty} (x-c)^2\, f_x(x)\ dx,$$

where $c = E(y)$. We found the marginal distribution of x in part (c) and the expected value of x in part (h), so in this case the variance is

$$V(x) = \int_0^1 (x-c)^2\left(\frac{12x^2+6x}{7}\right)\ dx.$$

To keep the calculation neater, let's leave the expected value as c for now, and we will plug in .714 at the end. The calculation proceeds as follows:

$$V(x) = \frac{6}{7}\int_0^1 (x^2 - 2cx + c^2)(2x^2 + x)\ dx$$

$$= \frac{6}{7}\int_0^1 2x^4 - 4cx^3 + 2c^2x^2 + x^3 - 2cx^2 + c^2x\ dx$$

$$= \frac{6}{7} \int_0^1 2x^4 - (4c-1)x^3 + 2c(c-1)x^2 + c^2 x \; dx$$

$$= \frac{6}{7} \left(\frac{2}{5}x^5 - \frac{4c-1}{4}x^4 + \frac{2c(c-1)}{3}x^3 + \frac{c^2}{2}x^2 \Big|_0^1 \right)$$

$$= \frac{6}{7} \left(\frac{2}{5} - \frac{4c-1}{4} + \frac{2c(c-1)}{3} + \frac{c^2}{2} \right)$$

$$= \frac{6}{7} \left(\frac{2}{5} - \frac{4(.714)-1}{4} + \frac{2(.714)(.714-1)}{3} + \frac{.714^2}{2} \right) = 0.047.$$

Finally, the standard deviation of x is simply the square root of the variance:

$$SD(x) = \sqrt{V(x)} = \sqrt{0.047} = 0.217.$$

(k) The variance of y is given by

$$V(y) = \int_{-\infty}^{\infty} (y-d)^2 \, f_y(y) \; dy,$$

where $d = E(y)$. We found the marginal distribution of y in part (d) and the expected value of y in part (i), so in this case the variance is

$$V(y) = \int_0^2 (y-d)^2 \left(\frac{4+3y}{14} \right) \; dy.$$

Again, to keep the calculation neater, let's leave the expected value as d for now, and we will plug in 1.14 at the end. The calculation proceeds as follows:

$$V(y) = \frac{1}{14} \int_0^2 (y^2 - 2dy + d^2)(4+3y) \; dy$$

$$= \frac{1}{14} \int_0^2 4y^2 - 8dy + 4d^2 + 3y^3 - 6dy^2 + 3d^2 y \; dy$$

$$= \frac{1}{14} \int_0^2 3y^3 + (4-6d)y^2 + (3d^2 - 8d)y + 4d^2 \; dy$$

$$= \frac{1}{14} \left(\frac{3}{4}y^4 + \frac{4-6d}{3}y^3 + \frac{3d^2-8d}{2}y^2 + 4d^2 y \Big|_0^2 \right)$$

$$= \frac{1}{14} \left(\frac{3}{4}(2)^4 + \frac{4-6d}{3}(2)^3 + \frac{3d^2-8d}{2}(2)^2 + 4d^2(2) \right)$$

$$= \frac{1}{14} \left(\frac{3}{4}(16) + \frac{4-6d}{3}(8) + \frac{3d^2-8d}{2}(4) + 8d^2 \right)$$

$$= \frac{1}{14} \left(12 + \frac{8(4-6d)}{3} + 2(3d^2 - 8d) + 8d^2 \right)$$

$$= \frac{1}{14} \left(12 + \frac{8[4-6(1.14)]}{3} + 2[3(1.14)^2 - 8(1.14)] + 8(1.14)^2 \right) = 0.313.$$

Finally, the standard deviation of y is simply the square root of the variance:

$$SD(y) = \sqrt{V(y)} = \sqrt{0.313} = 0.559.$$

(l) The covariance between x and y is given by

$$\text{Cov}(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(x-c)(y-d)\,f(x,y)\;dy\;dx,$$

where $c = E(x)$ and $d = E(y)$, which we will plug in only at the end of the calculation.[2] Substituting the bounds of x and y and their joint distribution gives us

$$\begin{aligned}
\text{Cov}(x,y) &= \int_0^1\int_0^2 (x-c)(y-d)\left[\frac{6}{7}\left(x^2+\frac{xy}{2}\right)\right]\;dy\;dx\\
&= \frac{6}{7}\int_0^1\int_0^2 (xy-dx-cy+cd)\left(x^2+\frac{xy}{2}\right)\;dy\;dx\\
&= \frac{6}{7}\int_0^1\int_0^2 x^3y-dx^3-cx^2y+cdx^2+\frac{x^2y^2}{2}-\frac{dx^2y}{2}-\frac{cxy^2}{2}+\frac{cdxy}{2}\;dy\;dx\\
&= \frac{6}{7}\int_0^1\left(\frac{x^3y^2}{2}-dx^3y-\frac{cx^2y^2}{2}+cdx^2y+\frac{x^2y^3}{6}-\frac{dx^2y^2}{4}-\frac{cxy^3}{6}+\frac{cdxy^2}{4}\Bigg|_0^2\right)\;dx\\
&= \frac{6}{7}\int_0^1\left(\frac{x^3(2)^2}{2}-dx^3(2)-\frac{cx^2(2)^2}{2}+cdx^2(2)+\frac{x^2(2)^3}{6}-\frac{dx^2(2)^2}{4}-\frac{cx(2)^3}{6}+\frac{cdx(2)^2}{4}\right)\;dx\\
&= \frac{6}{7}\int_0^1 2x^3-2dx^3-2cx^2+2cdx^2+\frac{4x^2}{3}-dx^2-\frac{4cx}{3}+cdx\;dx\\
&= \frac{6}{7}\left(\frac{x^4}{2}-\frac{dx^4}{2}-\frac{2cx^3}{3}+\frac{2cdx^3}{3}+\frac{4x^3}{9}-\frac{dx^3}{3}-\frac{2cx^2}{3}+\frac{cdx^2}{2}\Bigg|_0^1\right)\\
&= \frac{6}{7}\left(\frac{(1)^4}{2}-\frac{d(1)^4}{2}-\frac{2c(1)^3}{3}+\frac{2cd(1)^3}{3}+\frac{4(1)^3}{9}-\frac{d(1)^3}{3}-\frac{2c(1)^2}{3}+\frac{cd(1)^2}{2}\right)\\
&= \frac{6}{7}\left(\frac{1}{2}-\frac{d}{2}-\frac{2c}{3}+\frac{2cd}{3}+\frac{4}{9}-\frac{d}{3}-\frac{2c}{3}+\frac{cd}{2}\right)\\
&= \frac{6}{7}\left(\frac{1}{2}-\frac{1.14}{2}-\frac{2(.714)}{3}+\frac{2(.714)(1.14)}{3}+\frac{4}{9}-\frac{1.14}{3}-\frac{2(.714)}{3}+\frac{(.714)(1.14)}{2}\right)\\
&= -0.007.
\end{aligned}$$

(m) After all that work to calculate the marginal distributions, the expected values, the variances and standard deviations, and the covariance, all we have left to do to calculate the correlation between x and y is divide the covariance we calculated in part (l) by the standard deviations we calculated in parts (j) and (k):

$$\begin{aligned}
\text{Corr}(x,y) &= \frac{\text{Cov}(x,y)}{SD(x)SD(y)}\\
&= \frac{-0.007}{0.217\times 0.559} = -0.056.
\end{aligned}$$

[2] Please forgive the notational confusion here: d is used to denote both the expected value of y and the integration differential. The term dx appears inside the integrand as x times the expected value of y, and also as always at the end of the integral. Please note that every instance of dx inside the integrand refers to the former, not the latter.

10. (a) The correlation between two random variables is the covariance of the two variables divided by the product of their standard derivations. The standard deviations are the square roots of the variances:

$$\begin{aligned}\text{Corr}(X,Y) &= \frac{\text{Cov}(X,Y)}{SD(X)SD(Y)} \\ &= \frac{\text{Cov}(X,Y)}{\sqrt{V(X)}\sqrt{V(Y)}}.\end{aligned}$$

We know that $\text{Cov}(X,Y) = 5$, $V(X) = 4$, and $V(Y) = 9$ from the given information in this problem. So we simply plug these numbers in:

$$\text{Corr}(X,Y) = \frac{\text{Cov}(X,Y)}{\sqrt{V(X)}\sqrt{V(Y)}} = \frac{5}{\sqrt{4}\sqrt{9}} = \frac{5}{2 \times 3} = \frac{5}{6} = .833.$$

(b) We know from the problem that $A = 3X - 3Y + 7$. The variance of A is

$$V(A) = V(3X - 3Y + 7).$$

We can apply the rules of the variance of a sum from section 6.7 to find the value of this variance. First, adding a constant to a random variable does not change the variance of that random variable. So the variance becomes

$$V(A) = V(3X - 3Y).$$

Next, the variance of a weighted sum of two random variables is given by

$$V(aX + bY) = a^2V(X) + b^2V(Y) - 2ab\text{Cov}(X,Y),$$

which in this case is

$$\begin{aligned}V(3X - 3Y) &= (3)^2V(X) + (-3)^2V(Y) - 2(3)(-3)\text{Cov}(X,Y), \\ &= 9(4) + 9(9) + 18(5) = 207.\end{aligned}$$

(c) The problem tells us that $B = 5 - 2X + Y$. The goal is to calculate

$$V(B) = V(5 - 2X + Y).$$

We again use the rules for variances listed in section 6.7. The constant addend does not change the variance, so we can remove it,

$$V(B) = V(-2X + Y),$$

and the remaining expression is a weighted sum of random variables, subject to the rule

$$V(aX + bY) = a^2V(X) + b^2V(Y) - 2ab\text{Cov}(X,Y),$$

which in this case is

$$\begin{aligned}V(-2X + Y) &= (-2)^2V(X) + (1)^2V(Y) - 2(-2)(1)\text{Cov}(X,Y) \\ &= 4(4) + 9 + 4(5) = 45.\end{aligned}$$

(d) This problem asks us to calculate

$$\text{Cov}(A, B) = \text{Cov}\Big(3X - 3Y + 7,\, 5 - 2X + Y\Big)$$

Here we can apply the rules of covariances from section 7.4.4. First, adding a constant to either term of the covariance does not change the covariance, so we can remove both constant addends:

$$\text{Cov}(A, B) = \text{Cov}\Big(3X - 3Y,\, -2X + Y\Big).$$

Next we apply the rule for handling covariances of weighted sums of random variables,

$$\text{Cov}(aX + bY,\, cW + dZ) = ac\text{Cov}(X, W) + ad\text{Cov}(X, Z) + bc\text{Cov}(Y, W) + bd\text{Cov}(Y, Z),$$

which in this case is

$$\text{Cov}\Big(3X - 3Y,\, -2X + Y\Big) = (3)(-2)\text{Cov}(X, X) + (3)(1)\text{Cov}(X, Y) + (-3)(-2)\text{Cov}(Y, X) + (-3)(1)\text{Cov}(Y, Y).$$

$$= -6\text{Cov}(X, X) + 3\text{Cov}(X, Y) + 6\text{Cov}(Y, X) - 3\text{Cov}(Y, Y).$$

We also know that changing the order of the terms does not change the covariance, so we can combine the two middle covariances:

$$-6\text{Cov}(X, X) + 3\text{Cov}(X, Y) + 6\text{Cov}(X, Y) - 3\text{Cov}(Y, Y)$$

$$= -6\text{Cov}(X, X) + 9\text{Cov}(X, Y) - 3\text{Cov}(Y, Y).$$

Finally, we know that the covariance of a variable with itself is the variance. So our covariance becomes

$$-6V(X) + 9\text{Cov}(X, Y) - 3V(Y)$$

$$= -6(4) + 9(5) - 3(9) = -6.$$

(e) Now that we know the variance of A, the variance of B, and the covariance between them, we can plug these quantities into the formula for the correlation:

$$\text{Corr}(A, B) = \frac{\text{Cov}(A, B)}{SD(A)SD(B)}$$

$$= \frac{\text{Cov}(A, B)}{\sqrt{V(A)}\sqrt{V(B)}}$$

$$= \frac{-6}{\sqrt{207}\sqrt{45}} = -0.06.$$

Notice that the two transformations we applied, to generate A and B from X and Y, completely wiped out most of the correlation between these two variables, even though X and Y are themselves highly correlated.

11. (a) The first partial derivative of $E(y_i)$ with respect to x_{i1} is

$$\frac{\partial E(y_i)}{\partial x_{i1}} = \frac{\partial}{\partial x_{i1}}\Big(\alpha + \beta_1 x_{i1} + \beta_2 x_{i2} + \beta_3 x_{i2}^2 + \beta_4 x_{i2}^3 + \beta_5 x_{i3} + \beta_6 x_{i4} + \beta_7 x_{i3} x_{i4}\Big) = \beta_1.$$

(b) When the variable under consideration is assumed to have a strictly linear effect – no curvilinear terms or interactions – then the coefficient IS the first partial derivative. That means that the classic interpretation of a regression coefficient,
"a one-unit increase in x_{i1} is associated with a β_1 change in y_i, on average, after controlling for the other x variables in the model,"
is the interpretation of the first-partial derivative. To break this down further:

Classic regression interpretation	How it relates to partial derivatives
"A one-unit increase in x_{i1}"	This part comes from the fact that a derivative is a slope. When you calculate slope, you divide the rise over the run. The resulting slope is a change in y for a 1-unit change in x, simply because you can write any slope as a fraction of the slope over 1.
"is associated with"	"Associated" simply means we haven't taken steps to ensure this model is giving us true causation.
"a β_1 change in y_i,"	This also comes from the basic definition of a slope. Change in y over change in x.
"on average,"	This part refers to the fact that we are taking the derivative of the expected value of y_i instead of y_i itself.
"after controlling for the other x variables in the model."	This phrase refers to the fact that we are taking a *partial* derivative instead of a regular derivative. We are only considering the slope in one direction: the direction that refers to x_{i1}.

It is often useful to approach the interpretation of regression models by thinking about derivatives instead of coefficients.

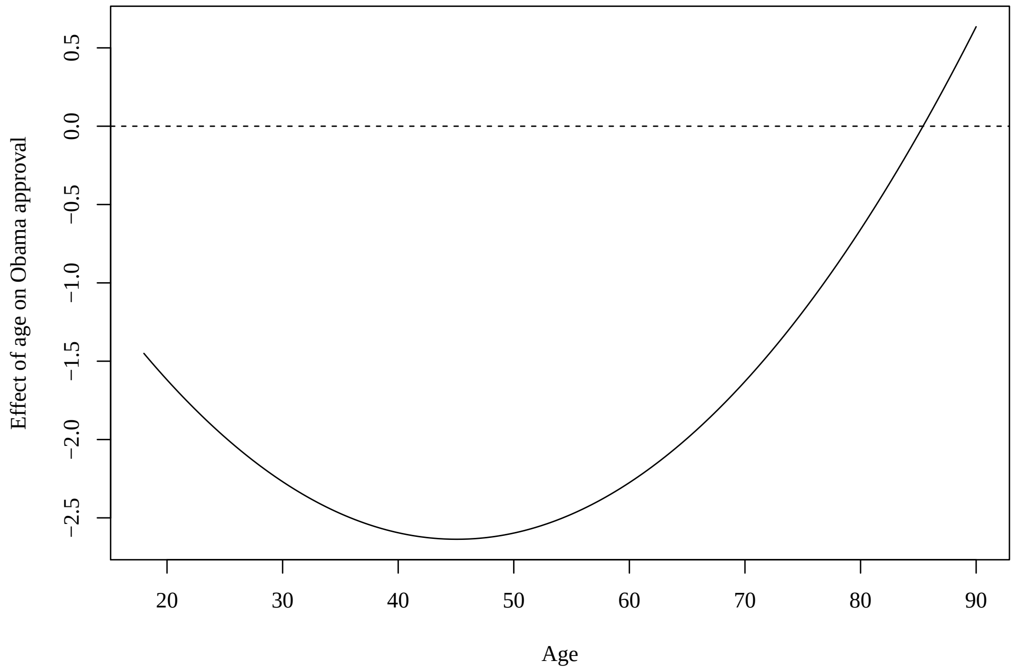
(c) The first partial derivative of $E(y_i)$ with respect to x_{i2} is

$$\frac{\partial E(y_i)}{\partial x_{i2}} = \frac{\partial}{\partial x_{i2}}\left(\alpha + \beta_1 x_{i1} + \beta_2 x_{i2} + \beta_3 x_{i2}^2 + \beta_4 x_{i2}^3 + \beta_5 x_{i3} + \beta_6 x_{i4} + \beta_7 x_{i3} x_{i4}\right) = \beta_2 + 2\beta_3 x_{i2} + 3\beta_4 x_{i2}^2.$$

(d) If $\beta_2 = 0.653, \beta_3 = -0.073$, and $\beta_4 = 0.00054$, then we can plug these values into the partial derivative:

$$\begin{aligned}\frac{\partial E(y_i)}{\partial x_{i2}} &= 0.653 + 2(-0.073)x_{i2} + 3(0.00054)x_{i2}^2 \\ &= 0.00162x_{i2}^2 - 0.146x_{i2} + 0.653.\end{aligned}$$

Remember that x_{i2} is age, so it is reasonable to plot this function over a domain of x_{i2} from 18 to 90. The graph is:



The graph says that age generally has a negative effect on a person's approval of Obama: that is, older people approve of Obama less. But this effect is most highly pronounced for middle-aged people, with the largest negative effect estimated for people who are about 45 years old. There is a smaller negative effect of age for younger people and for older people. People over 85 years old actually have a positive effect, although this effect is unlikely to be significantly different than zero. This graph would support a theory that says that people's political preferences change most in their 40s. In order to really draw these inferences, however, we would need to graph the 95% confidence interval around this partial derivative for every x.

(e) The first partial derivative of $E(y_i)$ with respect to x_{i4} is

$$\frac{\partial E(y_i)}{\partial x_{i4}} = \frac{\partial}{\partial x_{i4}}\left(\alpha + \beta_1 x_{i1} + \beta_2 x_{i2} + \beta_3 x_{i2}^2 + \beta_4 x_{i2}^3 + \beta_5 x_{i3} + \beta_6 x_{i4} + \beta_7 x_{i3} x_{i4}\right) = \beta_6 + \beta_7 x_{i3}.$$

(f) If we plug in $\beta_6 = 1$ and $\beta_7 = 3$ then this partial derivative becomes

$$\frac{\partial E(y_i)}{\partial x_{i4}} = 1 + 3x_{i3}.$$

Since x_{i3} is binary, we can break down this partial derivative into two cases. For men the partial derivative is $1 + 3(0) = 1$, and for women the partial derivative is $1 + 3(1) = 4$.

(g) That means that, in general, people who are more pro-choice are also more approving of Obama. For every unit more pro-choice a man is, he is on average 1 unit more favorable towards Obama. But for every unit more pro-choice a woman is, she is on average 4 units more favorable towards Obama. So this issue has a more dramatic effect on Obama approval for women than for men.

12. (a) We know from the problem that

$$\theta = \beta_1 x_1 + \beta_2 x_2 + \delta.$$

That means that the variance of θ is

$$V(\theta) = V(\beta_1 x_1 + \beta_2 x_2 + \delta),$$

where the we will treat the β coefficients as constants and the other terms as random variables. The formula for the variance of a weighted sum of three independent variables is

$$V(aX + bY + cZ) = a^2 V(X) + b^2 V(Y) + c^2 V(Z) + 2ab\text{Cov}(X,Y) + 2ac\text{Cov}(X,Z) + 2bc\text{Cov}(Y,Z).$$

We apply this formula to $V(\theta)$ to get

$$V(\beta_1 x_1 + \beta_2 x_2 + \delta) = \beta_1^2 V(x_1) + \beta_2^2 V(x_2) + V(\delta) + 2\beta_1\beta_2\text{Cov}(x_1,x_2) + 2\beta_1\text{Cov}(x_1,\delta) + 2\beta_2\text{Cov}(x_2,\delta).$$

We can directly measure $V(x_1)$, $V(x_2)$, and $\text{Cov}(x_1, x_2)$ from the data, and we assume that $\text{Cov}(x_1,\delta)$=0, $\text{Cov}(x_2,\delta)$=0, and $V(\delta) = 1$, so that this formula becomes

$$V(\beta_1 x_1 + \beta_2 x_2 + \delta) = \beta_1^2 V(x_1) + \beta_2^2 V(x_2) + 2\beta_1\beta_2\text{Cov}(x_1,x_2) + 1.$$

(b) We know that y_1 has the following regression equation:

$$y_1 = \delta_1\theta + \varepsilon_1.$$

To find the variance of y_1, we apply the formula for the variance of a weighted sum, again treating the coefficient λ as a constant:

$$\begin{aligned} V(y_1) &= V(\delta_1\theta + \varepsilon_1) \\ &= \delta_1^2 V(\theta) + V(\varepsilon_1) + 2\delta_1\text{Cov}(\theta,\varepsilon_1). \end{aligned}$$

We continue to assume that no error has a covariance with any other variable, so we can replace the covariance in this formula with zero,

$$V(y_1) = \delta_1^2 V(\theta) + V(\varepsilon_1),$$

we also assume that every error has a variance equal to 1,

$$V(y_1) = \delta_1^2 V(\theta) + 1.$$

We now replace $V(\theta)$ with the formula we derived in part (a):

$$V(y_1) = \delta_1^2\Big(\beta_1^2 V(x_1) + \beta_2^2 V(x_2) + 2\beta_1\beta_2\text{Cov}(x_1,x_2) + 1\Big) + 1.$$

(c) We can understand the covariance of x_1 and y_1 by substituting y_1's linear equation for y_1,

$$\mathrm{Cov}(x_1, y_1) = \mathrm{Cov}(x_1, \lambda_1\theta + \varepsilon_1),$$

and applying the rule of the covariance of a weighted sum,

$$\mathrm{Cov}(aX + bY,\, cW + dZ) = ac\mathrm{Cov}(X, W) + ad\mathrm{Cov}(X, Z) + bc\mathrm{Cov}(Y, W) + bd\mathrm{Cov}(Y, Z),$$

$$\mathrm{Cov}(x_1, \lambda_1\theta + \varepsilon_1) = \lambda_1\mathrm{Cov}(x_1, \theta) + \mathrm{Cov}(x_1, \varepsilon_1).$$

Once again, we assume that the errors are independent from all other variables, so that the covariance $\mathrm{Cov}(x_1, \varepsilon_1) = 0$, leaving us with

$$\lambda_1\mathrm{Cov}(x_1, \theta).$$

The problem now requires us to find this remaining covariance. First, we replace θ with its linear model,

$$\lambda_1\mathrm{Cov}(x_1, \beta_1 x_1 + \beta_2 x_2 + \delta),$$

and then we apply the rule regarding the covariances of weighted sums:

$$\lambda_1\mathrm{Cov}(x_1, \beta_1 x_1 + \beta_2 x_2 + \delta) = \lambda_1\Big(\beta_1\mathrm{Cov}(x_1, x_1) + \beta_2\mathrm{Cov}(x_1, x_2) + \mathrm{Cov}(x_1, \delta)\Big).$$

We can replace $\mathrm{Cov}(x_1, x_1)$ with $V(x_1)$, and as in part (a), we can directly measure $V(x_1)$, $V(x_2)$, and $\mathrm{Cov}(x_1, x_2)$ from the data, and we assume that $\mathrm{Cov}(x_1, \delta)$=0. The covariance is

$$\mathrm{Cov}(x_1, y_1) = \lambda_1\beta_1 V(x_1) + \lambda_1\beta_2\mathrm{Cov}(x_1, x_2).$$

The point of this exercise is to derive the formulas that underlie structural equation modeling. Most practitioners will consider these formulas to be too complex to try to understand. But now you know that all you need to derive these formulas is an understanding of covariance.