# Chapter 4: Limits and Derivatives

1. If you draw out a few regular polygons with more sides than 6, you will see that these shapes more and more closely resemble a circle as the number of sides increases. Therefore the answer to this question is: a circle.

   A regular polygon with even a huge number of sides is technically *not* a circle, although it looks like one. A regular polygon only becomes a true circle with *infinitely many* sides, a quantity that can only be understood with a limit. For similar reasons, it's impossible to display a true circle on a television or computer screen, just a close approximation of one, because even on high-resolution screens a close-up examination of the circle reveals that the image is composed of tiny squares.

   For more discussion about the connection between circles and the concept of infinity, and how this connection was used by Archimedes to derive the first close approximation of $\pi$, see the article "Take It to the Limit" by Steven Strogatz which appeared in the *New York Times* on April 4, 2010.

2. (a) $\lim\limits_{x\to 5} 2x^2 - 5x + 7$

   For this limit, nothing prevents us from simply plugging 5 into the function. Therefore the limit is

   $$2(5)^2 - 5(5) + 7 = 50 - 25 + 7 = 32.$$

   (b) $\lim\limits_{y\to\infty} \frac{1}{y^6}$

   If we plug in larger and larger values of $y$, so that these values approach infinity, the value of the function gets closer and closer to 0 without ever exactly reaching it. Therefore the value of the limit is 0.

   In general,

   $$\lim_{x\to\infty} \frac{1}{x^p} = 0$$

   as long as $p > 0$. We will use this property to solve some of the limits below.

   (c) $\lim\limits_{z\to 0} \frac{1}{z^6}$

   We cannot simply plug 0 into the function because that would mean we divide by zero. If we plug in values that approach 0 from the right (such as $z$=.1, .01, .00001) we see that the values of the function get larger and larger without bound, so the limit from the right is $\infty$. If we plug in values that approach 0 from the left (such as $z$=-.1, -.01, -.00001) we also see that the function approaches $\infty$ (which happens because the even exponent on $z$ causes negative values to become positive). Because the function approaches the same limit from the left and the right, this limit exists, and is equal to $\infty$.

(d) $\lim\limits_{x\to\infty} \dfrac{2x+3}{5x^2}$

We could plug in larger and larger values of $x$ and observe what happens to the function, but there's a more direct approach to solving this limit. Since there is addition in the numerator, we can break this function up into two fractions,

$$\lim_{x\to\infty} \frac{2x}{5x^2} + \frac{3}{5x^2}.$$

We can cancel a factor of $x$ from the top and bottom of the first fraction,

$$\lim_{x\to\infty} \frac{2}{5x} + \frac{3}{5x^2},$$

break up the limit over addition,

$$\lim_{x\to\infty} \frac{2}{5x} + \lim_{x\to\infty} \frac{3}{5x^2},$$

and bring the constant factors outside each limit,

$$\frac{2}{5} \lim_{x\to\infty} \frac{1}{x} + \frac{3}{5} \lim_{x\to\infty} \frac{1}{x^2}.$$

Both limits are equal to 0 according to the property we derived in part (b) (we can also see that quickly by plugging in large values of $x$ into each function). So the entire limit is

$$\frac{2}{5}(0) + \frac{3}{5}(0) = 0.$$

(e) $\lim\limits_{y\to\infty} \dfrac{3y^7 + 4y^6 - 2y^5 - 8y^3 - 7y + 1}{2y^7 + y^3 - 8}$

Here too, we can plug in larger and larger values of $y$ and observe what happens to the function, but there's a more direct approach to solving this limit. We can multiply the top and bottom of the fraction by the same thing, so let's multiply both sides by $1/y^7$:

$$\begin{aligned} &\lim_{y\to\infty} \frac{3y^7 + 4y^6 - 2y^5 - 8y^3 - 7y + 1}{2y^7 + y^3 - 8} \times \frac{\frac{1}{y^7}}{\frac{1}{y^7}} \\ &= \lim_{y\to\infty} \frac{3 + \frac{4}{y} - \frac{2}{y^2} - \frac{8}{y^4} - \frac{7}{y^6} + \frac{1}{y^7}}{2 + \frac{1}{y^4} - \frac{8}{y^7}}. \end{aligned}$$

According to the property we derived in part (b) the limit of each fraction approaches 0 as $y \to \infty$. Substituting 0 in for each of these fractions leaves us with a total limit of $\frac{3}{2}$.

(f) $\lim\limits_{z\to 3} \dfrac{z^2 - 5z + 6}{z - 3}$

We cannot simply plug in 3 as that would make the denominator equal to 0. But observe that the numerator can be factored. Two numbers that add to -5 and multiply to 6 are -2 and -3, so the numerator factors to

$$\lim_{z\to 3} \frac{(z-2)(z-3)}{z-3}.$$

Now a factor of $z - 3$ factors from the top and bottom, leaving us with a simpler limit in which we can plug in 3:

$$\lim_{z\to 3} z - 2 = 1.$$

(g) $\lim_{x \to 5+} \frac{1}{x-5}$

We cannot simply plug in 5, but this problem asks us to only consider the limit from the right. We can do that by plugging in values of the function that are closer and closer to 5, all of which are *greater than* 5. I plug several values, such as $x = 5.1, 5.01, 5.00001$, into the function and find that the function at these values is respectively 10, 100, 100000. The limit from the right is equal to $\infty$.

(h) $\lim_{y \to 7} \frac{12}{y-7}$

We cannot plug in 7, so let's consider the limits from the right and left. From the right, we plug in values such as $y = 7.1, 7.01, and 7.00001$ and observe that the function gets larger and larger, approaching $\infty$. From the left, we plug in values such as $y = 6.9, 6.99, and 6.99999$ and observe that the function gets smaller and smaller, approaching $-\infty$. Since the function approaches different limits from the left and the right, this limit does not exist.

(i) $\lim_{z \to \infty} \left(1 + \frac{1}{z}\right)^{2z}$

In order to solve this limit exactly, we employ a trick. Remember that the definition of the number $e$ is

$$e = \lim_{z \to \infty} \left(1 + \frac{1}{z}\right)^{z} \approx 2.718282\ldots$$

We can rewrite the function to reveal the definition of $e$ inside the limit. Remember the following property of exponents:

$$(x^a)^b = x^{ab}.$$

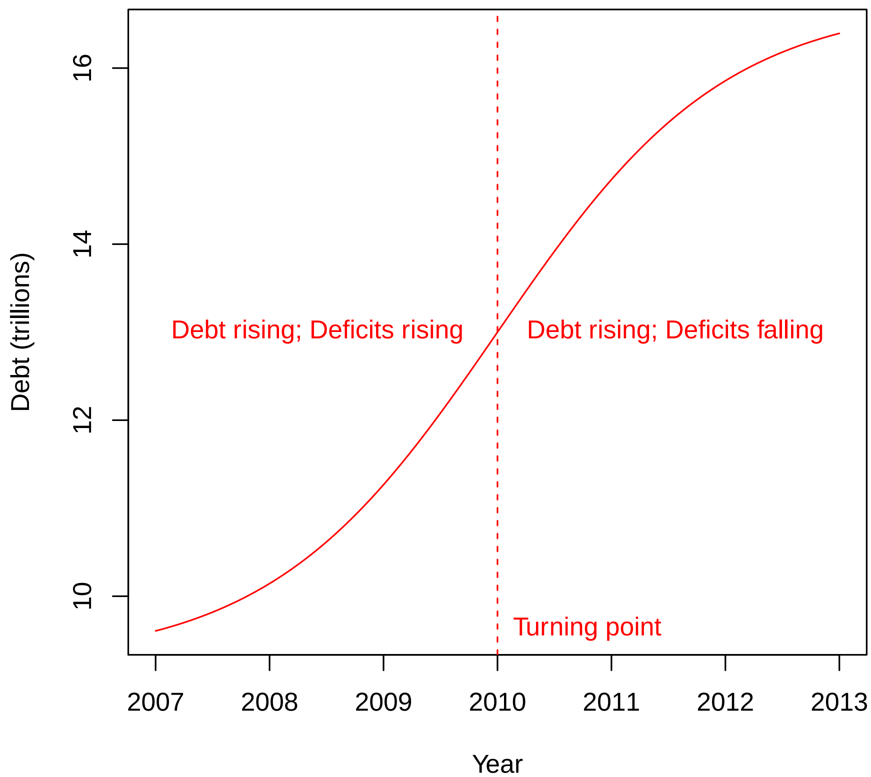
In other words, when an exponent contains two factors, we can rewrite that expression with two levels of exponents given by the two factors. In this case, we can rewrite

$$\begin{aligned} &\lim_{z \to \infty} \left(1 + \frac{1}{z}\right)^{2z} \\ &= \lim_{z \to \infty} \left(\left(1 + \frac{1}{z}\right)^{z}\right)^{2} \\ &= \lim_{z \to \infty} e^2 \approx 7.389056\ldots \end{aligned}$$

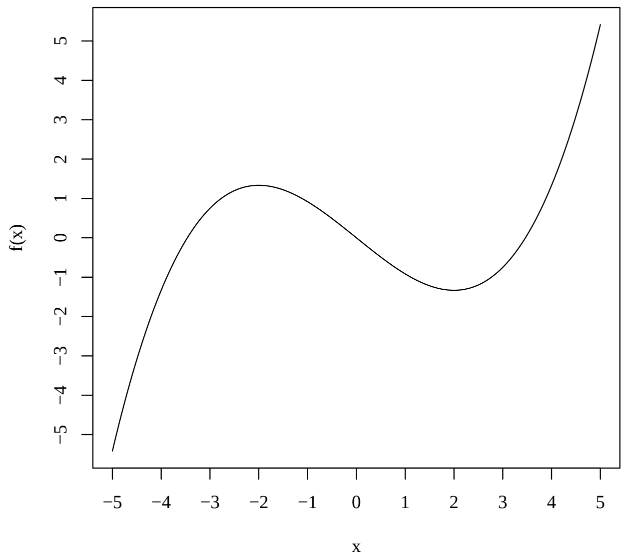
3. (a) If $t$ is the year, and $f(t)$ is the size of the national debt at year $t$, then $f'(t)$ is the *change in the debt from one year to the next*, which is just another way to phrase the definition of a deficit. To say that the deficit is changing is to refer to the derivative of the function for the deficit, which is itself the derivative of $f(t)$. So, "our deficits are falling at the fastest rate in 60 years" means that

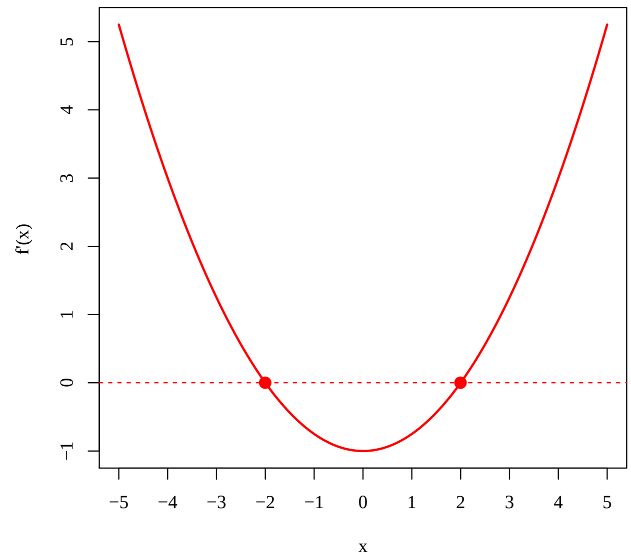
- $f''(t)$ is currently equal to a negative number (indicating that the deficit is *falling*), and
- that number is less than (more negative than) any number it has been for the last 60 years.

(b) Here is one example. Note that the graph is positive everywhere (we always have debt), and it's slope is positive everywhere (we always have deficits instead of surpluses), but the deficits are growing before 2010, and shrinking after 2010.




4. First, notice that the slope of the graph is zero at -2 (the top of the hill), and at 2 (the bottom of the valley). So we can start by plotting the points (-2,0) and (2,0) on the graph of the derivative. Next, observe that the slope is negative between $x = -2$ and $x = 2$, and positive everywhere else. Finally, note that the farther away the graph gets from $x = -2$ and $x = 2$, the the further the slope gets from 0. We can use these observations to construct the graph on the right below:




5. (a) $f(x) = x^6 + 5x^5 - 2x^2 + 8$

We begin by breaking the derivative up over addition and subtraction and bringing constant factors outside each derivative,

$$\begin{aligned} f'(x) &= \frac{d}{dx}\left(x^6 + 5x^5 - 2x^2 + 8\right) \\ &= \frac{d}{dx}(x^6) + \frac{d}{dx}(5x^5) - \frac{d}{dx}(2x^2) + \frac{d}{dx}(8), \\ &= \frac{d}{dx}(x^6) + 5\frac{d}{dx}(x^5) - 2\frac{d}{dx}(x^2) + \frac{d}{dx}(8). \end{aligned}$$

We apply the power rule of differentiation to the first three differentiation terms, and we replace the derivative of the constant in the fourth term with 0. The total derivative is

$$f'(x) = 6x^5 + 25x^4 - 4x.$$

(b) $g(y) = 3e^y - \sqrt{y}$

We begin by breaking the derivative up over addition and subtraction and bringing constant factors outside each derivative,

$$\begin{aligned} g'(y) &= \frac{d}{dy}\left(3e^y - \sqrt{y}\right) \\ &= \frac{d}{dy}(3e^y) - \frac{d}{dy}(\sqrt{y}) \\ &= 3\frac{d}{dy}(e^y) - \frac{d}{dy}(\sqrt{y}). \end{aligned}$$

According to the rules of differentiation in section 4.8, the derivative of the exponential function $e^y$ is simply $e^y$ again, and the derivative of $\sqrt{y}$ is $\frac{1}{2\sqrt{y}}$. Substituting these derivatives into the overall function gives us

$$g'(y) = 3e^y - \frac{1}{2\sqrt{y}}.$$

(c) $h(z) = \ln(z) + \dfrac{1}{z} + 3^z$

We break the derivative up over addition and subtraction,

$$h'(z) = \frac{d}{dz}\left(\ln(z) + \frac{1}{z} + 3^z\right)$$

$$= \frac{d}{dz}(\ln(z)) + \frac{d}{dz}(\frac{1}{z}) + \frac{d}{dz}(3^z),$$

and apply the rules of differentiation in section 4.8. The derivative of a natural logarithm is $\frac{1}{z}$, the derivative of the inverse function $\frac{1}{z}$ is $\frac{-1}{z^2}$, and the derivative of the exponential function $3^z$ is $\ln(3)3^z$. Substituting these derivatives into the overall function gives us

$$h'(z) = \frac{1}{z} - \frac{1}{z^2} + \ln(3)3^z.$$

(d) $j(x) = (x+3)^7(3x^4 - 2x^2 - 8)$

We could multiply this polynomial out, but that would be a lengthy process since there is an exponent of 7, and we would have to FOIL 7 times. Instead, let's apply the product rule. First, let $g(x) = (x+3)^7$ and let $h(x) = 3x^4 - 2x^2 - 8$. Then $j(x) = g(x)h(x)$, and the product rule for derivatives tells us that

$$j'(x) = g'(x)h(x) + h'(x)g(x).$$

The derivative of the first function technically requires the chain rule:

$$g(x) = A^7, \qquad A = x + 3$$

$$\frac{dg}{dx} = \frac{dg}{dA}\frac{dA}{dx}$$

$$= 7A^6 \times 1 = 7(x+3)^6.$$

The derivative of the second function is

$$\frac{dh}{dx} = \frac{d}{dx}\left(3x^4 - 2x^2 - 8\right)$$

$$= 3\frac{d}{dx}(x^4) - 2\frac{d}{dx}(x^2) - \frac{d}{dx}(8),$$

since derivatives break up across addition and subtraction, and since constant factors can be brought outside a derivative. Then this derivative is

$$12x^3 - 4x.$$

Plugging both functions and both derivatives into the product rule, we get

$$j'(x) = \left(7(x+3)^6\right)\left(3x^4 - 2x^2 - 8\right) + \left((x+3)^7\right)\left(12x^3 - 4x\right).$$

We could simplify this derivative, but this is a correct answer, so let's stop here.

(e) $k(y) = e^{\sqrt{y}}$

This function has layers, so it requires the chain rule. Let's rewrite the function as

$$k(y) = e^A, \qquad A = \sqrt{y}.$$

The derivative of the outer layer is just

$$\frac{d}{dA}\left(e^A\right) = e^A = e^{\sqrt{y}},$$

and the derivative of the inner layer is

$$\frac{d}{dy}\left(\sqrt{y}\right) = \frac{1}{2\sqrt{y}}.$$

Multiplying the derivatives of both layers together, we get

$$k'(y) = \frac{e^{\sqrt{y}}}{2\sqrt{y}}.$$

(f) $l(z) = \dfrac{\ln(z)}{z}$

This function has a quotient, so let's set $f(z) = \ln(z)$ (high) and $g(z) = z$ (low), so that the quotient rule ("low d-high minus high d-low over low-squared") is

$$l'(z) = \frac{d}{dz}\left(\frac{f(z)}{g(z)}\right) = \frac{g(z)f'(z) - f(z)g'(z)}{g(z)^2}.$$

The derivative of $f(z)$ is

$$f'(z) = \frac{d}{dz}\left(\ln(z)\right) = \frac{1}{z}.$$

The derivative of $g(z)$ is

$$g'(z) = \frac{d}{dz}\left(z\right) = 1.$$

Plugging both functions and their derivatives into the quotient rule, we get

$$l'(z) = \frac{z(\frac{1}{z}) - \ln(z)}{z^2} = \frac{1 - \ln(z)}{z^2}.$$

(g) $m(x) = \dfrac{1}{1+e^{-x}}$

This function has layers, so let's apply the chain rule. First we rewrite the function as

$$m(x) = \frac{1}{A}, \quad A = 1 + e^{B}, \quad B = -x.$$

The derivative of the outermost layer is

$$\frac{d}{dA}\left(\frac{1}{A}\right) = \frac{-1}{A^2} = \frac{-1}{(1+e^{B})^2} = \frac{-1}{(1+e^{-x})^2}.$$

The derivative of the middle layer is

$$\frac{d}{dB}\left(1+e^{B}\right) = e^{B} = e^{-x}.$$

And the derivative of the innermost layer is

$$\frac{d}{dx}(-x) = -1.$$

Multiplying the layers together, we get

$$m'(x) = \frac{-1}{(1+e^{-x})^2} \times e^{-x} \times -1 = \frac{e^{-x}}{(1+e^{-x})^2}.$$

(h) $n(y) = \sqrt{y}e^{\sqrt{y}}$

This derivative requires the product rule. Let $f(y) = \sqrt{y}$ and let $g(y) = e^{\sqrt{y}}$, so that the derivative is given by

$$n'(y) = f'(y)g(y) + g'(y)f(y).$$

First, the derivative of $f(y)$ is

$$\frac{d}{dy}(\sqrt{y}) = \frac{1}{2\sqrt{y}}.$$

Next, the derivative of $g(y)$ requires the chain rule. We rewrite $g(y)$ as

$$g(y) = e^{A}, \qquad A = \sqrt{P},$$

we take the derivative of the outer layer,

$$\frac{d}{dA}\left(e^{A}\right) = e^{A} = e^{\sqrt{y}},$$

we take the derivative of the inner layer,

$$\frac{d}{dy}(\sqrt{y}) = \frac{1}{2\sqrt{y}},$$

and we multiply the layers together

$$g'(y) = \frac{e^{\sqrt{y}}}{2\sqrt{y}}.$$

Now that we have $f'(y)$ and $g'(y)$, we plug these derivatives into the product rule:

$$\begin{aligned} n'(y) &= \left(\frac{1}{2\sqrt{y}}\right)\left(e^{\sqrt{y}}\right) + \left(\frac{e^{\sqrt{y}}}{2\sqrt{y}}\right)\left(\sqrt{y}\right) \\ &= (\sqrt{y}+1)\frac{e^{\sqrt{y}}}{2\sqrt{y}}. \end{aligned}$$

(i) $p(z) = \dfrac{e^{z^2+4}}{\ln(z)}$

We apply the quotient rule, with $f(z) = e^{z^2+4}$ and $g(z) = \ln(z)$ so that

$$p'(z) = \frac{d}{dz}\left(\frac{f(z)}{g(z)}\right) = \frac{g(z)f'(z) - f(z)g'(z)}{g(z)^2}.$$

The derivative of $f(z)$ requires the chain rule, so we rewrite the function as

$$f(z) = e^A, \qquad A = z^2 + 4.$$

The derivative of the outer layer is

$$\frac{d}{dA}\left(e^A\right) = e^A = e^{z^2+4},$$

and the derivative of the inner layer is

$$\frac{d}{dz}(z^2 + 4) = 2z,$$

so the entire derivative of $f(z)$ is

$$f'(z) = 2ze^{z^2+4}.$$

The derivative of $g(z)$ is

$$\frac{d}{dz}\left(\ln(z)\right) = \frac{1}{z}.$$

Substituting these functions and their derivatives into the quotient rule, we get

$$p'(z) = \frac{\ln(z)2re^{z^2+4} - \frac{e^{z^2+4}}{z}}{\ln(z)^2}.$$

(j) $q(x) = \ln(x^3 + 2x)$

We can rewrite the function as

$$q(x) = \ln(A), \qquad A = x^3 + 2x.$$

The chain rule tells us that

$$\frac{df}{dx} = \frac{df}{dA}\frac{dA}{dx}.$$

$$\frac{df}{dA} = \frac{d}{dA}\left(\ln(A)\right) = \frac{1}{A} = \frac{1}{x^3 + 2x},$$

$$\frac{dA}{dx} = \frac{d}{dx}(x^3 + 2x) = 3x^2 + 2.$$

$$\frac{df}{dx} = \frac{df}{dA}\frac{dA}{dx} = \frac{1}{x^3 + 2x} \times (3x^2 + 2)$$

$$q'(x) = \frac{3x^2 + 2}{x^3 + 2x}.$$

(k) $r(y) = e^{1/(y^2+2y-2)}$

We can rewrite the function with the following layers:

$$r(x) = e^A, \quad A = \tfrac{1}{B}, \quad B = y^2 + 2y - 2.$$

$$\frac{dm}{dA} = \frac{d}{dA}\left(e^A\right) = e^A = e^{1/(y^2+2y-2)}.$$

$$\frac{dA}{dB} = \frac{d}{dB}\left(\frac{1}{B}\right) = \frac{-1}{B^2} = \frac{-1}{(y^2+2y-2)^2}.$$

$$\frac{dB}{dy} = \frac{d}{dy}\left(y^2 + 2y - 2\right) = 2y - 2.$$

$$\frac{dm}{dy} = \frac{dm}{dA}\frac{dA}{dB}\frac{dB}{dy} = e^{1/(y^2+2y-2)} \times \frac{-1}{(y^2+2y-2)^2} \times (2y-2).$$

$$r'(y) = \frac{-(2y-2)e^{1/(y^2+2y-2)}}{(y^2+2y-2)^2}.$$

No need to simplify any further.

(l) $s(z) = \ln(z^3 + 2z)e^{(1/z^2+2z-2)}$

Notice that this is the product of the functions from parts (a) and (b), so we can write

$$s(z) = f(z)g(z).$$

By the product rule of derivatives, we know that

$$s'(z) = f(z)g'(z) + f'(z)g(z).$$

We've already found each of these terms, so we can simply plug them in:

$$s'(z) = \left(\ln(z^3+2z)\right)\left(\frac{-(2z-2)e^{1/(z^2+2z-2)}}{(z^2+2z-2)^2}\right) + \left(\frac{3z^2+2}{z^3+2z}\right)\left(e^{1/(z^2+2z-2)}\right).$$

(m) $t(x) = \dfrac{\sqrt{x^2+3}}{x}$

This function is the quotient of two functions, so we apply the quotient rule of differentiation. In this case,

$$t(x) = \frac{f(x)}{g(x)},$$

where

$$f(x) = \sqrt{x^2+3}, \quad \text{and} \quad g(x) = x.$$

The derivative of $f(x)$ requires the chain rule. Set $f(x) = \sqrt{A}$ and $A = x^2 + 3$. Then

$$f'(x) = \frac{1}{2\sqrt{A}} \times 2x = \frac{2x}{2\sqrt{x^2+3}} = \frac{x}{\sqrt{x^2+3}}.$$

The derivative of $g(x) = x$ is just $g'(x) = 1$. Plugging these terms into the formula for the quotient rule:

$$t'(x) = \frac{\frac{x^2}{\sqrt{x^2+3}} - \sqrt{x^2+3}}{x^2}.$$

(n) $v(y) = \sqrt{\dfrac{(y^4 - 3y^2)\ln(7y-4)}{e^{y^3-2y}}}$

This is an advanced chain rule problem, but the steps are the same as always. First we write the function in terms of its layers:

$$\begin{aligned} v(y) &= \sqrt{A}, & A &= \frac{BC}{D}, \\ B &= y^4 - 3y^2, & C &= \ln(E), \\ D &= e^F, & E &= 7y - 4, \\ F &= y^3 - 2y. \end{aligned}$$

Next we take the derivative of each layer. The derivative of the outermost layer is

$$v'(y) = \frac{1}{2\sqrt{A}}.$$

The trickiest part here is the derivative of $A$, which involves *both* the product rule and the quotient rule! Let's consider this derivative first. Note that $A$ is a quotient of functions, so we can first apply the quotient rule:

$$A' = \frac{D(BC)' - (BC)D'}{D^2}.$$

This expression contains the derivative of the product of two functions $(BC)'$, for which we apply the product rule:

$$A' = \frac{D(B'C + C'B) - (BC)D'}{D^2}.$$

Since this step involved the quotient and product rules, we **stop here** and multiply the derivatives together:

$$v'(y) = \frac{1}{2\sqrt{A}} \frac{D(B'C + C'B) - (BC)D'}{D^2}.$$

The problem is now to find $B'$, $C'$ and $D'$ and substitute them into this function. The derivative of $B$ is straightforward:

$$B' = 4y^3 - 6y.$$

The derivative of $C$ itself requires that we use the chain rule:

$$C' = \frac{1}{E}E' = \frac{7}{E}.$$

The derivative of $D$ also requires the chain rule:

$$D' = e^F F' = (3y^2 - 2)e^F.$$

Substituting $B'$, $C'$, and $D'$ into the overall derivative gives us

$$v'(y) = \frac{1}{2\sqrt{A}} \frac{D\left((4y^3-6y)C + (\frac{7}{E})B\right) - (BC)(3y^2-2)e^F}{D^2}.$$

Now we substitute back in for the capital letters, starting with $A$,

$$v'(y) = \frac{1}{2\sqrt{\frac{BC}{D}}} \frac{D\left((4y^3-6y)C + (\frac{7}{E})B\right) - (BC)(3y^2-2)e^F}{D^2},$$

then $B$,

$$v'(y) = \frac{1}{2\sqrt{\frac{(y^4-3y^2)C}{D}}} \frac{D\left((4y^3-6y)C + (\frac{7}{E})(y^4-3y^2)\right) - (y^4-3y^2)C(3y^2-2)e^F}{D^2},$$

then $C$,

$$v'(y) = \frac{1}{2\sqrt{\frac{(y^4-3y^2)\ln(E)}{D}}} \frac{D\left((4y^3-6y)\ln(E) + (\frac{7}{E})(y^4-3y^2)\right) - (y^4-3y^2)\ln(E)(3y^2-2)e^F}{D^2},$$

then $D$,

$$v'(y) = \frac{1}{2\sqrt{\frac{(y^4-3y^2)\ln(E)}{e^F}}} \frac{e^F\left((4y^3-6y)\ln(E) + (\frac{7}{E})(y^4-3y^2)\right) - (y^4-3y^2)\ln(E)(3y^2-2)e^F}{e^{2F}},$$

then $E$,

$$v'(y) = \frac{1}{2\sqrt{\frac{(y^4-3y^2)\ln(7y-4)}{e^F}}} \frac{e^F\left((4y^3-6y)\ln(7y-4) + (\frac{7}{7y-4})(y^4-3y^2)\right) - (y^4-3y^2)\ln(7y-4)(3y^2-2)e^F}{e^{2F}},$$

and finally $F$,

$$v'(y) = \frac{1}{2\sqrt{\frac{(y^4-3y^2)\ln(7y-4)}{e^{y^3-2y}}}} \frac{e^{y^3-2y}\left((4y^3-6y)\ln(7y-4) + (\frac{7}{7y-4})(y^4-3y^2)\right) - (y^4-3y^2)\ln(7y-4)(3y^2-2)e^{y^3-2y}}{e^{2y^3-4y}}.$$

This is about as difficult a derivative as anyone will have to take by hand. But we've not shied away from the hard problems and we won't start now. The answer isn't pretty, but if you follow and understand the steps we need to take this derivative, then you have a strong grasp of differentiation.

6. (a) The first derivative requires the product rule:

$$f'(x) = x\frac{d}{dx}\Big(\ln(x)\Big) + \ln(x)\frac{d}{dx}\Big(x\Big) = x\frac{1}{x} + \ln(x) = 1 + \ln(x).$$

The second derivative is

$$f''(x) = \frac{d}{dx}\Big(1 + \ln(x)\Big) = \frac{1}{x}.$$

The third derivative is

$$f'''(x) = \frac{d}{dx}\left(\frac{1}{x}\right) = \frac{-1}{x^2}.$$

The 4th derivative is

$$f^{(4)}(x) = \frac{d}{dx}\left(\frac{-1}{x^2}\right) = \frac{d}{dx}\left(-x^{-2}\right) = 2x^{-3} = \frac{2}{x^3}.$$

The 5th derivative is

$$f^{(5)}(x) = \frac{d}{dx}\left(\frac{2}{x^3}\right) = \frac{d}{dx}\left(2x^{-3}\right) = -6x^{-4} = \frac{-6}{x^4}.$$

The 6th derivative is

$$f^{(6)}(x) = \frac{d}{dx}\left(\frac{-6}{x^4}\right) = \frac{d}{dx}\left(-6x^{-4}\right) = 24x^{-5} = \frac{24}{x^5}.$$

Are you beginning to see the pattern? For the $n$th derivative, there's a fraction. The numerator is $(n-2)!$ (where the ! mark denotes a factorial, the product of all positive whole numbers from 1 to $n-2$ – for a definition of a factorial see section 3.3.2) and the denominator is $x^{n-1}$. But also, the fraction is multiplied by -1 for odd derivatives, and 1 by even derivatives, which we can represent as $(-1)^n$. So overall, a formula for the $n$th derivative is

$$f^{(n)}(x) = (-1)^n \frac{(n-2)!}{x^{n-1}}.$$

(b) The first derivative requires the product rule:

$$g'(x) = x\frac{d}{dx}\left(e^x\right) + e^x\frac{d}{dx}\left(x\right) = xe^x + e^x = (x+1)e^x.$$

The second derivative is

$$g''(x) = (x+1)\frac{d}{dx}\left(e^x\right) + e^x\frac{d}{dx}\left(x+1\right) = (x+1)e^x + e^x = (x+2)e^x.$$

The third derivative is

$$g'''(x) = (x+2)\frac{d}{dx}\left(e^x\right) + e^x\frac{d}{dx}\left(x+2\right) = (x+2)e^x + e^x = (x+3)e^x.$$

So it appears that a formula for the $n$th derivative is

$$g^{(n)}(x) = (x+n)e^x.$$

7. (a) At this point, you are able to quickly see that the derivative is $f'(x) = 3x^2$. But why is this true? The limit definitions relate to the basic concept that a derivative represents: the instantaneous rate of change of a function. Using the first limit definition, the derivative is

$$\begin{aligned} f'(a) &= \lim_{x\to a}\frac{f(x)-f(a)}{x-a} \\ &= \lim_{x\to a}\frac{x^3-a^3}{x-a}. \end{aligned}$$

The numerator factors using the difference of cubes ("SOAP") formula from section 1.7.2:

$$f'(a) = \lim_{x\to a} \frac{(x-a)(x^2+ax+a^2)}{x-a},$$

$$= \lim_{x\to a} x^2 + ax + a^2.$$

Since we've removed $x-a$ from the denominator, it's now safe to plug in $a$ for $x$:

$$f'(a) = a^2 + a(a) + a^2 = 3a^2.$$

Using the second limit definition, the derivative is

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h\to 0} \frac{(x+h)^3 - x^3}{h}$$

$$= \lim_{h\to 0} \frac{x^3 + 3x^2h + 3xh^2 + h^3 - x^3}{h}$$

$$= \lim_{h\to 0} \frac{3x^2h + 3xh^2 + h^3}{h}$$

$$= \lim_{h\to 0} \frac{h(3x^2 + 3xh + h^2)}{h}$$

$$= \lim_{h\to 0} 3x^2 + 3xh + h^2.$$

Since we've removed $h$ from the denominator, it's now safe to plug in 0 for $h$:

$$f'(x) = 3x^2 + 3x(0) + (0)^2 = 3x^2.$$

(b) The derivative of $g(y) = y^2 + 2y + 8$ is $g'(y) = 2y + 2$. To prove that this function really is the derivative of $g(y)$, we can use the first limit definition of the derivative:

$$g'(a) = \lim_{y\to a} \frac{f(y)-f(a)}{y-a}$$

$$= \lim_{y\to a} \frac{y^2+2y+8-(a^2+2a+8)}{y-a}$$

$$= \lim_{y\to a} \frac{y^2-a^2+2y-2a+8-8}{y-a}$$

$$= \lim_{y\to a} \frac{(y^2-a^2)+(2y-2a)}{y-a}$$

$$= \lim_{y\to a} \frac{(y-a)(y+a)+2(y-a)}{y-a}$$

$$= \lim_{y\to a} (y+a) + 2 = (a+a) + 2 = 2a + 2.$$

Using the second limit definition:

$$g'(y) = \lim_{h\to 0} \frac{g(y+h)-g(y)}{h}$$

$$= \lim_{h\to 0} \frac{(y+h)^2 + 2(y+h) + 8 - (y^2+2y+8)}{h}$$

$$
\begin{aligned}
&= \lim_{h\to 0} \frac{y^2+2yh+h^2+2y+2h+8-y^2-2y-8}{h} \\
&= \lim_{h\to 0} \frac{y^2-y^2+2yh+h^2+2y-2y+2h+8-8}{h} \\
&= \lim_{h\to 0} \frac{2yh+h^2+2h}{h} \\
&= \lim_{h\to 0} \frac{h(2y+h+2)}{h} \\
&= \lim_{h\to 0} 2y+h+2 = 2y+2.
\end{aligned}
$$

8. (a) The first derivative of the normal distribution requires the chain rule. I rewrite the function as

$$f(x) = \frac{1}{\sqrt{2\pi}} e^A, \qquad A = -\frac{x^2}{2}.$$

Then the derivative is

$$\frac{df}{dx} = \frac{df}{dA}\frac{dA}{dx} = \left(\frac{1}{\sqrt{2\pi}}e^A\right)\left(-x\right) = \frac{-x}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}.$$

The second derivative requires the product rule. If we let $g(x) = -x$, then the first derivative can be written as

$$f'(x) = \left(-x\right)\left(\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}\right) = g(x)f(x).$$

Note that the second factor in this particular problem is the original function $f(x)$. So the second derivative is

$$
\begin{aligned}
f''(x) &= g'(x)f(x) + g(x)f'(x) \\
&= (-1)\left(\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}\right) + (-x)\left(\frac{-x}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}\right) \\
&= \left(\frac{x^2}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}\right) - \left(\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}\right). \\
&= (x^2-1)\left(\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}\right).
\end{aligned}
$$

If we plug in 0 into the original function, we get

$$f(0) = \left(\frac{1}{\sqrt{2\pi}}e^{-\frac{0^2}{2}}\right) = \frac{1}{\sqrt{2\pi}}.$$

If we plug 0 into the first derivative, we get

$$f'(0) = \left(\frac{-0}{\sqrt{2\pi}}e^{-\frac{0^2}{2}}\right) = 0.$$

If we plug 0 into the second derivative, we get

$$f''(0) = (0^2-1)\left(\frac{1}{\sqrt{2\pi}}e^{-\frac{0^2}{2}}\right) = \frac{-1}{\sqrt{2\pi}}.$$

So the second-order Taylor approximation is the polynomial:

$$
\begin{aligned}
f(x) &\approx \frac{-1}{\sqrt{2\pi}}\frac{x^2}{2} + 0x + \frac{1}{\sqrt{2\pi}} \\
&= \frac{-1}{\sqrt{8\pi}}x^2 + \frac{1}{\sqrt{2\pi}}.
\end{aligned}
$$

(b) I graphed the normal distribution and its Taylor approximation using Stata, but any program that is capable of graphing can do the trick. First, type

```
gen x = 6*(_n/_N) - 3
```

The statement `_n` refers to each particular observation number, and the statement `_N` refers to the total number of observations. Type "browse" in the command window to see that this command produces an equally spaced sequence of numbers between -3 and 3. This variable will be used as the independent variable for both the normal distribution and the Taylor approximation. Next, get the $y$ values of the normal distribution by typing

```
gen norm = normalden(x)
```

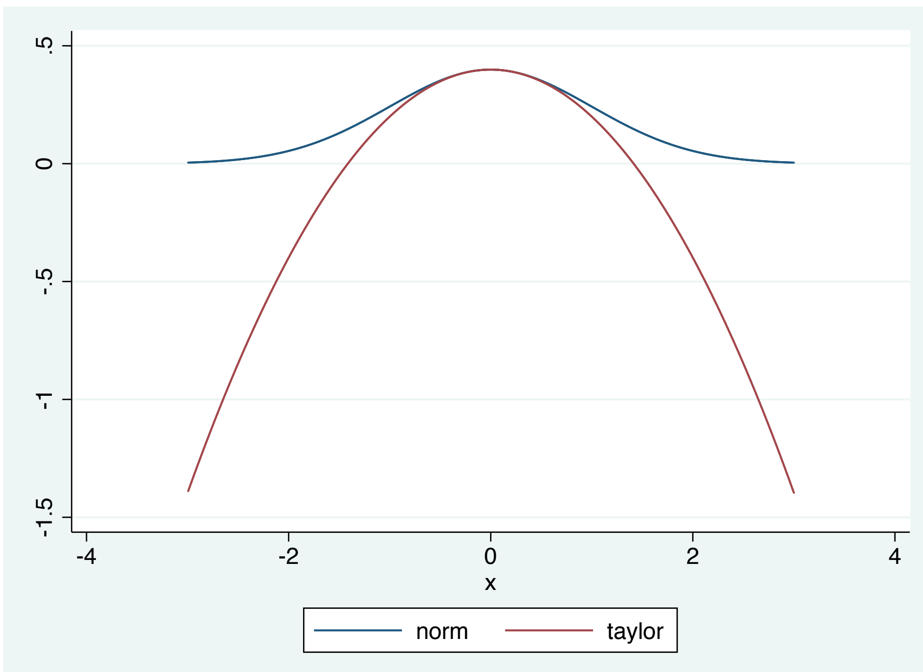
and get the $y$ values of the Taylor approximation by typing

```
gen taylor = (-1/sqrt(8*c(pi)))*x^2 + (1/sqrt(2*c(pi)))
```

where I ignore `x` since it's coefficient is 0. Finally, to produce the graph, type

```
twoway(line norm x)(line taylor x)
```

Here is the graph I obtain:




We approximate the normal distribution well around the centering point 0, and continue to approximate well out to about 1 and -1. The quality of the approximation declines rapidly for points further than 1 away from 0. If we estimated more derivatives though, we could obtain a higher-order Taylor polynomial which would conform to the normal distribution for points farther out.

9. (a) The log-likelihood is the natural logarithm of the likelihood function, which in this case is

$$\ell(p|y_1, \ldots, y_N) = \ln\left(\prod_{i=1}^{N} p^{y_i}(1-p)^{1-y_i}\right).$$

To simplify, we first remember that $\ln(ab) = \ln(a) + \ln(b)$, that a logarithm turns multiplication inside into addition outside. The same rule applies to long-products: a logarithm turns a long-product inside into a summation outside. So the log-likelihood simplifies to

$$\ell(p|y_1,\ldots,y_N) = \sum_{i=1}^{N} \ln\left(p^{y_i}(1-p)^{1-y_i}\right).$$

Next the logarithm breaks up the multiplication inside:

$$\ell(p|y_1,\ldots,y_N) = \sum_{i=1}^{N} \left[\ln\left(p^{y_i}\right) + \ln\left((1-p)^{1-y_i}\right)\right].$$

Now the exponents inside each logarithm can be brought down as factors:

$$\ell(p|y_1,\ldots,y_N) = \sum_{i=1}^{N} \Big[y_i \ln(p) + (1-y_i)\ln(1-p)\Big].$$

The summation can be broken up over addition,

$$\ell(p|y_1,\ldots,y_N) = \sum_{i=1}^{N} y_i \ln(p) + \sum_{i=1}^{N}(1-y_i)\ln(1-p),$$

and any factor that does not contain the index $i$ can be brought outside the summation. In this case, such factors are the logged probabilities:

$$\ell(p|y_1,\ldots,y_N) = \ln(p)\sum_{i=1}^{N} y_i + \ln(1-p)\sum_{i=1}^{N}(1-y_i).$$

Finally, the last summation can be broken up over subtraction,

$$\ell(p|y_1,\ldots,y_N) = \ln(p)\sum_{i=1}^{N} y_i + \ln(1-p)\left(\sum_{i=1}^{N}(1) - \sum_{i=1}^{N} y_i\right),$$

and the sum of 1s, repeated $N$ times, is just $N$:

$$\ell(p|y_1,\ldots,y_N) = \ln(p)\sum_{i=1}^{N} y_i + \ln(1-p)\left(N - \sum_{i=1}^{N} y_i\right).$$

(b) We are taking the derivative of the log-likelihood function, treating $p$ as the independent variable and the $y$ terms as constants. The problem is

$$\frac{d\ell}{dp} = \frac{d}{dp}\left[\ln(p)\sum_{i=1}^{N} y_i + \ln(1-p)\left(N - \sum_{i=1}^{N} y_i\right)\right].$$

The first step is to remember that a derivative breaks up over addition. Since a summation is just repeated addition, the derivative of a summation is the summation of the derivative of each term:

$$\frac{d\ell}{dp} = \frac{d}{dp}\left[\ln(p)\sum_{i=1}^{N} y_i\right] + \frac{d}{dp}\left[\ln(1-p)\left(N - \sum_{i=1}^{N} y_i\right)\right].$$

Here we treat $p$ as the independent variable and the $y_i$ terms as constants. We next bring the constant factors out of each derivative:

$$\frac{d\ell}{dp} = \left(\sum_{i=1}^{N} y_i\right)\frac{d}{dp}\Big(\ln(p)\Big) + \left(N - \sum_{i=1}^{N} y_i\right)\frac{d}{dp}\Big(\ln(1-p)\Big).$$

We've reduced the problem to taking the derivative of two logarithms. The second one requires the chain rule:

$$\frac{d}{dp}\ln(p) = \frac{1}{p},$$

$$\frac{d}{dp}\ln(1-p) = \frac{1}{1-p}(1-p)' = \frac{-1}{1-p}.$$

Substituting these terms into the derivative we get

$$\begin{aligned}\frac{d\ell}{dp} &= \left(\sum_{i=1}^{N} y_i\right)\frac{1}{p} + \left(N - \sum_{i=1}^{N} y_i\right)\frac{-1}{1-p} \\ &= \frac{\sum_{i=1}^{N} y_i}{p} - \frac{N - \sum_{i=1}^{N} y_i}{1-p}.\end{aligned}$$

(c) The problem is to solve

$$\frac{\sum_{i=1}^{N} y_i}{p} - \frac{N - \sum_{i=1}^{N} y_i}{1-p} = 0$$

for $p$. First we can bring one term over to the other side,

$$\frac{\sum_{i=1}^{N} y_i}{p} = \frac{N - \sum_{i=1}^{N} y_i}{1-p},$$

cross-multiply,

$$(1-p)\sum_{i=1}^{N} y_i = p\left(N - \sum_{i=1}^{N} y_i\right),$$

and distribute,

$$\sum_{i=1}^{N} y_i - p\sum_{i=1}^{N} y_i = pN - p\sum_{i=1}^{N} y_i.$$

We cancel $p\sum y_i$ from both sides,

$$\sum_{i=1}^{N} y{=}pN,$$

and solve for $p$ by dividing by $N$:

$$p = \frac{\sum_{i=1}^{N} y_i}{N}.$$

Substantively, this answer is the average of all the observed 0s and 1s. It makes perfect sense. If we survey 1000 people, and 600 say they voted for the incumbent, then there is a $\frac{600}{1000} = 0.6$ probability of voting for the incumbent.

(d) All we have to do is plug -3, 0 and 3 into

$$p_i = \frac{1}{1 + e^{-(0.2+0.5x_i)}}.$$

These calculations are

$$p_i(-3) = \frac{1}{1 + e^{-(0.2+0.5(-3))}} = 0.21,$$

$$p_i(0) = \frac{1}{1 + e^{-(0.2+0.5(0))}} = 0.55,$$

$$p_i(3) = \frac{1}{1 + e^{-(0.2+0.5(3))}} = 0.85.$$

So a very liberal voter has a 0.21 probability of voting for the incumbent, a moderate voter has a 0.55 probability of voting for the incumbent (indicating that the incumbent has won the moderate voters), and a very conservative voter has a 0.85 probability of voting for the incumbent.

(e) The problem is to take the derivative of

$$p_i = \frac{1}{1 + e^{-(0.2+0.5x_i)}}.$$

This derivative requires the chain rule. We break the function into layers as follows:

$$p_i = \frac{1}{A}, \quad A = 1 + e^B, \quad B = -(0.2 + 0.5x_i).$$

The derivatives of the layers are

$$p_i' = \frac{-1}{A^2}, \quad A' = e^B, \quad B' = -0.5.$$

No derivative involves the quotient or product rule. So the next step is to multiply the layers' derivatives together:

$$\frac{dp_i}{dx_i} = \frac{0.5e^B}{A^2}.$$

We substitute for $A$,

$$\frac{dp_i}{dx_i} = \frac{0.5e^B}{(1 + e^B)^2},$$

and we substitute for $B$:

$$\frac{dp_i}{dx_i} = \frac{0.5e^{-(0.2+0.5x_i)}}{(1 + e^{-(0.2+0.5x_i)})^2}.$$

This formula expresses the marginal change in the probability of voting for the incumbent for any ideology $x$. So for a moderate voter, the change in probability is

$$\frac{dp_i}{dx_i}(0) = \frac{0.5e^{-(0.2+0.5(0))}}{(1 + e^{-(0.2+0.5(0)i)})^2} = \frac{0.5e^{-0.2}}{(1 + e^{-0.2})^2} = 0.124.$$

(f) Remember that a derivative tells us the *instantaneous slope* of a function at a particular point. So the instantaneous slope of the probability of voting for the incumbent is 0.124 for moderate voters. But what does this really mean? Slope is the change in $y$ for a change in $x$. We consider the change in $x$ to be 1, as in change in miles per ONE hour. So, when $x$ is 0, a one-unit increase in $x$ is associated with an increase in $y$ of about 0.124.

That's not exactly the same thing as saying that moving from moderate ($x = 0$) to to slightly conservative ($x = 1$) is associated with a 0.124 increase in the probability of voting for the incumbent. It's possible that this instantaneous slope is slightly different at $x = 1$ than it is as $x = 0$, which would alter this difference. But exactly at $x = 0$, the rate of change of the dependent variable for a unit increase in $x$ is 0.124.