APPENDIX

Introduction to concrete manipulative materials

Manipulative materials are any materials that allow pupils to physically touch, move and rearrange them. In mathematics, they can model operations on numbers as well as the numbers themselves, and so allow learners to explore ideas, patterns and relationships in a concrete, rather than an abstract, way. Concrete materials are multisensory in that they can be appreciated by sight and by touch. They promote learning by visual, spatial and kinaesthetic routes. Teachers who ensure that all work with concrete materials is accompanied by lots of talk and discussion also cater for the auditory route to learning.

There is a variety of concrete material available to maths teachers these days, but too much variety can create problems. Presenting new models to illustrate new procedures can leave pupils with an incoherent view of maths as a series of isolated topics. For example, using a spike abacus for demonstrations of place value but for nothing else encourages pupils to compartmentalise place value thinking quite separately from thinking about mental calculations. Another common example is teaching division as repeated subtraction, but later explaining fractions through shading pictures of pizza slices, which does nothing to help pupils see the interconnection between the two concepts. This kind of fragmentation is particularly detrimental to learners with dyscalculia or those with little number sense or natural ‘feel’ for numbers. Pupils with difficulties benefit from having the kind of coherent model that highlights the patterns and connections within the field of mathematics.

In my view, by far the best and most versatile apparatus to use with pupils who experience difficulties with maths are continuous base-10 materials such as Cuisenaire rods and Dienes blocks. They are the most robust materials, in the sense of being capable of modelling many different situations and procedures at many different levels. Naturally, for very young children discrete materials such as counters, nuggets or cubes will precede work with rods or blocks, but overuse of discrete material tends to encourage pupils to cling to inefficient counting-in-ones strategies. Many of the teaching suggestions in this book are designed to take pupils beyond such immature strategies. Working with the right concrete materials and explicitly building connections between topics helps to foster a cohesive view of mathematics as a rational subject whose components are interrelated and interdependent.

Counters

Counters can be any small discrete objects. I particularly like the nuggets made of coloured or iridescent glass that Dorian Yeo introduced me to, and that are sold as flower vase fillers and table decorations, but I also occasionally use the smallest rods (1 cm cubes) from a Cuisenaire rods set as counters.

Counters are very useful for work with smaller numbers but are not suitable for numbers much above 20.

Counters are particularly useful for arranging and rearranging into dot patterns to model the numbers up to 10. In my own teaching, I use the dice patterns for the numbers 1 to 6 and doubles or near-double patterns for the numbers 7 to 10. These kinds of distinctive visual patterns allow the learner to recognise and comprehend each number as a whole, so that, say, the familiar dice pattern for 5 can be instantly read as 5 without any need to count the items. By extension, a pattern showing two 5s side by side can be easily read as 10, again without any need to count. Being able to visualise, and therefore know for sure, one fact about each number provides pupils with a secure starting point from which to derive new facts through logic and reasoning.

Numicon

Numicon is a system in which flat plastic rectangular shapes contain circular holes into which round pegs can be fitted, the number of holes corresponding to the number that is being represented. It is a popular system in many schools, partly because it is supported by so many ready-made resources.

Numicon provides separate boards for each of the numbers up to 10 with the holes regularly spaced and arranged in two parallel rows, such that all the even numbers are rectangles, while all the odd numbers have a protrusion creating an L-shape at one end. The problem with this organisation of numbers into pairs is that the larger numbers are difficult to distinguish without counting the holes. For example, 7 must be read as 2  2  2  1 and is therefore easily confused with 9 which is presented as 2  2  2  2  1. Similarly, the numbers 6, 8 and 10 are easily confused. A more significant problem is that I find the static shapes of the Numicon materials unhelpful for exploring number bonds: although they are perfect for showing the difference between odd and even numbers, they are much less successful at revealing the various ways in which numbers can be built out of, and split back into, number components.

The Numicon board-and-peg system, confined as it is to the numbers up to 10, is no substitute for base-10 materials such as Cuisenaire rods or Dienes. Unlike base-10 equipment, Numicon numbers are made of discrete items (i.e. the number 5, for example, is presented as five ones), discrete materials being far more limited in scope than continuous materials (in which the number 5, for example, is presented as one five). However, Numicon can offer a possible substitute for dot patterns: both are systems for organising quantities from 1 to 10 into a particular, fixed arrangement. Like dot patterns, Numicon pegs become cumbersome and impractical for numbers much beyond 20, and in both systems there is a danger of promoting counting strategies because the nature of the equipment emphasises the fact that all numbers are presented as a collection of ones. The limitation of discrete materials has obviously been recognised by Numicon, who now include Cuisenaire rods (which they call number rods) amongst their recommended apparatus. Because the number rods that Numicon sell are identical in every way to the rods invented by Cuisenaire, every suggestion that I make in this book that involves Cuisenaire rods can be performed, in exactly the same way, with rods supplied by Numicon.

Cuisenaire rods

Cuisenaire rods were invented by the Belgian educator Georges Cuisenaire in the 1940s to teach arithmetic to primary school children. They consist of a set of rods, made of wood or plastic, each with a cross-section measuring 1 cm square. The ten different lengths of rods start at 1 cm long and increase by increments of 1 cm. Although the largest rod is the orange 10-rod, Cuisenaire rods are compatible with Dienes blocks which provide for larger denominations. Each size of Cuisenaire rod has its own distinctive and unvarying colour, which makes its length easy to identify without the need for measuring it against unit cubes and counting the units. This is a compellingly significant characteristic and is, indeed, the main advantage of using continuous materials, rather than discrete objects. Because the rods do not carry labels or unit markings, they encourage quantities to be seen as a whole, rather than a collection of single units. This, in turn, encourages the development of efficient calculation methods that do not depend on counting in ones.

Cuisenaire rods became very popular in the 1950s and 1960s, and were considered at the time to be a turning point in the teaching of elementary mathematics. Their unfortunate lapse into disuse had more to do with the logistical difficulties of classroom management than to any uncertainty about their intrinsic educational value, when used correctly.

I have used Cuisenaire rods in my own teaching for many years. As an aid to parents who want to support their children at home, I have produced a small leaflet of ideas that you can download from the OR  (print both pages back-to-back on a single A4 sheet of paper and fold). I can warmly recommend Professor Mahesh Sharma’s various publications as well as his teacher training videos (available in the UK from Berkshire Mathematics, and in the USA from the Center for Teaching & Learning Mathematics, in Framingham, Massachusetts). Other information on using Cuisenaire rods can be obtained from the Cuisenaire Rod Company in the UK, or hand2mind (previously ETA) in the USA.

Dienes blocks

Zoltan Dienes produced his ‘attribute blocks’ not long after Cuisenaire produced his rods. The idea has been copied many times since then and the blocks are now often known generically as ‘base-10 materials’. They consist of wooden or plastic blocks, all of the same colour, and are based on 1 cm cubes formed into single cubes (1), longs (10) and square flats (100). Base-10 sets nowadays also offer large cubes to represent 1000. Unlike Cuisenaire rods, base-10 materials have scored surfaces to highlight the 1 cm cube units from which they are built. Unlike the rods, there are no blocks to represent the numbers between 1 and 10, which means that the numbers below 10 have to be represented by discrete cubes and counted out one by one. For this reason, I prefer to use Cuisenaire rods for numbers up to 100, supplemented by the larger Dienes blocks for 3-digit numbers.

Stern materials

Stern materials were developed by Dr Catherine Stern at roughly the same time as Cuisenaire rods and are very similar in structure and purpose. Stern blocks are based on 2 cm cubes, rather than Cuisenaire’s 1 cm cross-section, which makes them satisfyingly chunky for small hands and for those with dyspraxia who may find the 1 cm rods too fiddly. Stern blocks are deliberately notched, so that they look like a series of cubes stuck together, which inevitably means that many children will count the cubes in order to find the length of the longer blocks. In order to minimise counting, each length has a fixed colour. Unfortunately, the same colours do not represent the same numbers on Stern blocks as on Cuisenaire rods, making the two systems incompatible despite their conceptual similarities.

There are some very useful and beautifully made pieces of equipment designed to go with the Stern blocks. This partly explains why Stern is so much more expensive than either Cuisenaire or Dienes. I particularly like the number boxes, especially the 10-box, the 20-tray, the number track and the dual board. Less successful in my view are the pattern boards, which were the inspiration for Numicon, in which quantities up to 10 are arranged in pairs.

Every suggestion that I make in this book that involves Cuisenaire rods can be performed just as well, and in exactly the same way, with Stern blocks. The Stern blocks’ only disadvantage – apart from the notched surfaces showing all numbers as a collection of ones, as already mentioned – is that their scale does not allow them to be combined with base-10 blocks. This limits their use to early numeracy work and to numbers below 100.

Hybrid materials

Bead strings, Unifix cubes and various types of abacus all attempt to bridge the divide between discrete and continuous materials. They are all useful in their own way, though none is as versatile as the combination of Cuisenaire rods and Dienes blocks.

Of the various lengths and types of bead strings, I have found the most useful to be a string of ten beads, with a colour change after five beads, because it models the all-important complement pairs that add up to 10.

Of the different kinds of abacus, I much prefer the Slavonic abacus, which is arranged as a field of 100 beads with a colour change after five beads and after five rows of beads. The colour change allows pupils to read quantities with a minimum of counting in ones. Ideas for using a Slavonic abacus with pupils who have difficulties with maths can be found in Eva Grauberg’s work.

How to use concrete materials

Concrete materials should be carefully introduced and demonstrated by the teacher, who should explain to pupils what they are for, namely to make a connection between numerical magnitude and the abstract symbols we use to record numerical quantities, to make maths principles visible, and to help learners develop insight and intuition. Appropriate concrete materials allow learners to make meaning for themselves and to create a model for understanding maths that they can internalise. This cognitive model supports pupils’ progression to abstract high-level thinking.

Concrete materials are not intended to be a primitive alternative to calculating machines and should never be used in a mechanical way, simply to find an answer.

Whenever using concrete materials, it pays to be sensitive to the danger of the counting trap and to be aware of the potential for unintentionally reinforcing bad habits or inefficient calculation strategies. It is, for example, extremely important to use concrete materials in a way that discourages pupils from persistently counting in ones. For this reason, in my own teaching I always prefer: counters arranged in predictable dice or domino patterns rather than random patterns or fixed patterns that are insufficiently visually distinctive; the smooth surfaces of the coloured Cuisenaire rods as opposed to the notched or scored blocks belonging to the Stern and Dienes materials; whole rods to represent the numbers up to 10 rather than rows or collections of single cubes each representing the number 1; the Slavonic abacus in preference to either a spike abacus or an abacus with a different colour for each row of beads; bead strings that mimic the Slavonic abacus’ layout by a change in colour after every group of five in preference to bead strings constructed of many different colours of beads or with ten consecutive beads all of the same colour; number tracks that mimic the Slavonic abacus’ layout by a change in colour after every group of five spaces in preference to numbered tracks.

Concrete materials should not be used only for demonstration purposes, nor should they be used only for very basic work. They are for pupils to handle and explore, and they are most useful when the same materials are used at different stages, for different topics and at different levels of difficulty. As teachers, we must keep reminding our pupils that the actual mathematics is not what happens to numerals on paper, but what happens to numbers that are subjected to mathematical operations. Paper and pencil are just useful ways to record what happens, or to support our memory while we engage in mental calculation and abstract thinking.

After using concrete materials for a particular topic, encourage pupils to make simple sketches to represent or record what they have just done. The pictorial or diagrammatic stage is a very important transition between concrete and abstract work.